Unit - I
Random Variables

Probability of an event :

$$P(A) = \frac{\text{favourable cases}}{\text{Possible cases}}$$

Random experiment :

All outcomes are known, but we can't predict the exact outcome.

Trial

Performing an experiment.

Sample Space :

All possible outcomes of an experiment

eg 1) Tossing a coin 2) rolling a die

$S = \{H, T\}$ $S = \{1, 2, 3, 4, 5, 6\}$

Event :

Subset of a Sample Space.

eg In the toss of a coin, let A be the event of getting head.

Equally Likely.

Cannot be expected to happen in Preferrence to any other.

eg Turning up of the head or tail is equally likely.

Mutually Exclusive:

Occurrence of one of them does not Prevent the occurrence of others.

eg Either head or tail will turn up. Both cannot happen at the same time.

Exhaustive Events:

A set is exhaustive if it includes all possible outcomes of a trial.

Axioms of Probability:

Let S be a Sample Space. To each event A, there is a number $P(A)$ associated, Probability of A satisfying the following conditions

(i) $P(A) \geq 0$

(ii) $P(S) = 1$

(iii) If $A_1, A_2, \ldots A_n$ are mutually exclusive events, then

Addition theorem :

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Conditional Probability :

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) \neq 0$$

Multiplication theorem:

$$P(A \cap B) = P(A) \cdot P(B/A)$$

Independent events

$$P(A \cap B) = P(A) \cdot P(B).$$

Random Variables :

It is a function X which assigns a number to every outcome of a random experiment.

eg Tossing two unbiased coins.

Outcomes : HH, HT, TH, TT

Random Variable X : No. of heads.

(Assigning real no's) : (2, 1, 1, 0)

Mathematical defn : $X : S \longrightarrow R$

① A random Variable X has the following Probability distribution

x	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find (i) the value of k

(ii) $P[1.5 < X < 4.5 / X > 2]$

(iii) the Smallest Value of λ for which $P[X \leq \lambda] > \frac{1}{2}$.

Soln

(i) WKT $\sum P_i = 1$

$\Rightarrow 0+k+2k+2k+3k+k^2+2k^2+7k^2+k = 1$

$9k+10k^2 = 1$

$10k^2+9k-1 = 0$

$(k+1)(10k-1) = 0$

$k=-1 \quad k = 1/10$

(ii) $P[1.5 < X < 4.5 / X > 2] = \dfrac{P[(1.5<X<4.5)\cap(X>2)]}{P[X>2]}$

$= \dfrac{P[2<X<4.5]}{P[X>2)} = \dfrac{P[X=3]+P[X=4]}{P[X>2]}$

$= \dfrac{\frac{2}{10}+\frac{3}{10}}{\frac{6}{10}+\frac{10}{100}} = \dfrac{5}{7}$

(iii) $P[X \leq \lambda] > 1/2$

$\Rightarrow P[X \leq 3] = 1/2$

(2) A random variable X takes the values 1, 2, 3 & 4 such that $2P[X=1] = 3P[X=2] = P[X=3] = 5P[X=4]$. Find the Probability distribution and cumulative distribution function of X.

Soln

Let

$$2P[X=1] = 3P[X=2] = P[X=3] = 5P[X=4] = k$$

$$\left.\begin{aligned} P[X=1] &= k/2 \\ P[X=2] &= k/3 \\ P[X=3] &= k \\ P[X=4] &= k/5 \end{aligned}\right\} \text{——— (1)}$$

WKT $\sum p_i = 1$

$$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$

$$\frac{15k + 10k + 30k + 6k}{30} = 1$$

$$\frac{61k}{30} = 1$$

$$\boxed{k = \frac{30}{61}}$$

Sub $k = \frac{30}{61}$ in (1)

$$P[X=1] = \frac{30}{61} \times \frac{1}{2} = \frac{15}{61}$$

$$P[X=2] = \frac{30}{61} \times \frac{1}{3} = \frac{10}{61}$$

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i) Probability distribution function is

X	1	2	3	4
$P(x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

ii) Cummulative distribution function is

X	$F[x] = P[X \le x]$
1	$\frac{15}{61}$
2	$\frac{15}{61} + \frac{10}{61} = \frac{25}{61}$
3	$\frac{15}{61} + \frac{10}{61} + \frac{30}{61} = \frac{55}{61}$
4	$\frac{15}{61} + \frac{10}{61} + \frac{30}{61} + \frac{6}{61} = \frac{61}{61} = 1$

(3) A R.V X has the following probability function

x	0	1	2	3	4	5	6	7	8
$P(x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

i) Determine a

ii) Evaluate $P(X<3)$, $P(X \ge 4)$, $P(0 < X \le 5)$

iii) Find the distribution function of X

Soln

WKT $\sum p(x) = 1$

x	0	1	2	3	4	5	6	7	8
$P(x)$	$1/81$	$3/81$	$5/81$	$7/81$	$9/81$	$11/81$	$13/81$	$15/81$	$17/81$
$F[x]$	$1/81$	$4/81$	$9/81$	$\frac{16}{81}$	$\frac{25}{81}$	$\frac{36}{81}$	$\frac{49}{81}$	$\frac{64}{81}$	1.

$$P(X<3) = P(0) + P(1) + P(2)$$

$$= \frac{1}{81} + \frac{3}{81} + \frac{5}{81} = \frac{9}{81}$$

$$P(X \geq 4) = P(4) + P(5) + P(6) + P(7) + P(8)$$

$$= \frac{9}{81} + \frac{11}{81} + \frac{13}{81} + \frac{15}{81} + \frac{17}{81}$$

$$= \frac{65}{81}$$

$$P(0 < X \leq 5) = P(X = 1, 2, 3, 4, 5)$$

$$= \frac{3}{81} + \frac{5}{81} + \frac{7}{81} + \frac{9}{81} + \frac{11}{81}$$

$$= \frac{25}{81}$$

4) The probability Mass function of a R.V X is defined as $P(X=0) = 3c^2$, $P(X=1) = 4c - 10c^2$, $P(X=2) = 5c - 1$ where $c > 0$ and $P(X=r) = 0$ if $r = 0, 1, 2$, find

i) the value of c

(iii) The distribution function of X.

(iv) The Largest value of X for which $F[x] < 1/2$

Soln

x	0	1	2
$P(x)$	$3c^2$	$4c-10c^2$	$5c-1$

i) To find c

$$\Sigma p(x) = 1$$

$$3c^2 + 4c - 10c^2 + 5c - 1 = 1$$

$$-7c^2 + 9c - 2 = 0$$

$$7c^2 - 9c + 2 = 0$$

$$(c-1)(7c-2) = 0$$

$$c = 1, 2/7$$

$c \neq 1$.

$\therefore \boxed{c = 2/7}$

(iv)

x	0	1	2
$P(x)$	$\frac{12}{49}$	$\frac{16}{49}$	$\frac{3}{7}$
$F[x]$	$\frac{12}{49}$	$\frac{28}{49}$	1

(ii) $P(0 < X < 2 / X > 0) = \dfrac{P[X=1] \cap P[X=1,2]}{P[X=1,2]}$

$$= \frac{P[X=1]}{P[X=1,2]} = \frac{\frac{16}{49}}{\frac{16}{49}+\frac{3}{7}}$$

$$= \frac{16}{49} \times \frac{49}{37}$$

$$= 16/37$$

(iii) $F[x] < 1/2$ is 0.

Continuous Random Variable:

X takes all its possible values in an interval.

Probability density function:

Let X be a continuous R.V then a function $f(x)$ is called pdf if

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Cummulative distribution function

$$F[x] = P[X \leq x] = \int^{x} f(x)\,dx.$$

Note:

i) $f(x) = \frac{d}{dx} F[x]$

(ii) If X is continuous, then $P[a<x<b] = F[b]-F[a]$

1) Find the value of c given that pdf of a r.v X as $f(x) = \frac{c}{x^3}$, $1 < x < \infty$

Soln

W.k.T $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{1}^{\infty} \frac{c}{x^3}\,dx = 1$$

$$c\int_{1}^{\infty} x^{-3}\,dx = 1$$

$$c\left[\frac{x^{-2}}{-2}\right]_1^{\infty} = 1$$

$$\frac{c}{-2}[0-1] = 1$$

$$\boxed{c=2}$$

2) If X is a discrete r.v taking the values 1, 2, 3, ... with probability function $P[X=x] = \frac{c^x}{x!}$, $x = 1, 2, \ldots$ then find the value of c.

Soln

$$\sum p_i = 1$$

$$\left[\frac{c^1}{1!}+\frac{c^2}{2!}+\frac{c^3}{3!}+\cdots\right]=1$$

$$\left[e^c-1\right]=1 \qquad \left[\because e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right]$$

$$e^c=2$$

$$\boxed{c=\log 2}$$

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(5) If the pdf of a continuous r.v X is given by $f(x)=\begin{cases} ax, & 0\le x\le 1 \\ a, & 1\le x\le 2 \\ 3a-ax, & 2\le x\le 3 \\ 0, & \text{elsewhere}\end{cases}$

i) find the value of a

ii) find the cdf of X

iii) If x_1, x_2, x_3 are 3 independent observations of X, what is the probability that exactly one of these 3 is greater than 1.5?

Soln

i) To find a.

WKT $\int_{-\infty}^{\infty} f(x)\,dx=1$

$$\int_0^1 ax\,dx+\int_1^2 a\,dx+\int_2^3 (3a-ax)\,dx=1$$

$$a\left[\frac{x^2}{2}\right]_0^1+a\left[x\right]_1^2+\left[3ax-\frac{ax^2}{2}\right]_2^3=1$$

$$a\left[\tfrac{1}{2}-0\right]+a\left[2-1\right]+\left[9a-\frac{9a}{2}-6a+2a\right]=1$$

$$\frac{a}{2} + a + 5a - \frac{9a}{2} = 1$$

$$\frac{4a}{2} = 1$$

$$\boxed{a = \frac{1}{2}}$$

$$\therefore f(x) = \begin{cases} \frac{x}{2}, & 0 \le x \le 1 \\ \frac{1}{2}, & 1 \le x \le 2 \\ \frac{1}{2}(3-x), & 2 \le x \le 3 \\ 0, & \text{Otherwise} \end{cases}$$

ii) Cummulative distribution function of X is

$$F[x] = P[X \le x]$$

$$= \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{3}^{x} f(x)\,dx$$

Case (i) $x \le 0$

$$F[x] = \int_{-\infty}^{0} f(x)\,dx = 0$$

Case (ii) $0 \le x \le 1$

$$F[x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} \left(\frac{x}{2}\right) dx$$

$$= \left[0 + \frac{x^2}{1}\right]^{x} = \frac{x^2}{1.}$$

case (iii) $1 \le x \le 2$

$$F[x] = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{x} \frac{1}{2}\,dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{1}{2}x\right]_1^x$$

$$= \left[\frac{1}{4} - 0\right] + \left[\frac{x}{2} - \frac{1}{2}\right]$$

$$= \frac{1}{4} + \frac{x}{2} - \frac{1}{2}$$

$$= \frac{x}{2} - \frac{1}{4}$$

Case (iv) $2 \le x \le 3$

$$F[x] = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{2} \frac{1}{2}\,dx + \int_{2}^{x} \frac{1}{2}(3-x)\,dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{x}{2}\right]_1^2 + \frac{1}{2}\left[3x - \frac{x^2}{2}\right]_2^x$$

$$= \left[\frac{1}{4} - 0\right] + \left[\frac{2}{2} - \frac{1}{2}\right] + \frac{1}{2}\left[\left(3x - \frac{x^2}{2}\right) - \left(6 - \frac{4}{2}\right)\right]$$

$$= \frac{1}{4} + \left[1 - \frac{1}{2}\right] + \frac{1}{2}\left[\left(3x - \frac{x^2}{2}\right) - \frac{8}{2}\right]$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{3x}{2} - \frac{x^2}{4} - 2$$

$$= \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}$$

Case (v) $x \geq 3$

$$F[x] = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{x} f(x)\,dx$$

$$F[x] = 1$$

$$F[x] = \begin{cases} 0, & x \leq 0 \\ \frac{x^2}{4}, & 0 \leq x \leq 1 \\ \frac{x}{2} - \frac{1}{4}, & 1 \leq x \leq 2 \\ \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}, & 2 \leq x \leq 3 \\ 1, & x > 3. \end{cases}$$

(iii) $P(1 \leq x \leq 2.5) = \int_{1}^{2.5} f(x)\,dx$

$$= \int_{1}^{2} f(x)\,dx + \int_{2}^{2.5} f(x)\,dx$$

$$= \int_{1}^{2} \frac{1}{2}\,dx + \int_{2}^{2.5} \frac{1}{2}(3-x)\,dx$$

$$= \left[\frac{x}{2}\right]_{1}^{2} + \frac{1}{2}\left[3x - \frac{x^2}{2}\right]_{2}^{2.5}$$

$$= \left[1 - \frac{1}{2}\right] + \frac{1}{2}\left[\left(3(2.5) - \frac{5}{2}\right) - \left(6 - \frac{4}{2}\right)\right]$$

$$= \frac{1}{2} + \frac{1}{2}\left[\left(7.5 - \frac{5}{2}\right] - \left(\frac{8}{2}\right)\right]$$

$$= \frac{11}{16}.$$

iv $P(X > 1.5) = \int_{1.5}^{\infty} f(x)\,dx$

$= \int_{1.5}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx$

$= \int_{1.5}^{2} \frac{1}{2}\,dx + \int_{2}^{3} \frac{1}{2}(3-x)\,dx + 0$

$= \frac{1}{2}[x]_{1.5}^{2} + \frac{1}{2}\left[3x - \frac{x^2}{2}\right]_{2}^{3}$

$= \frac{1}{2}\left[2 - \frac{3}{2}\right] + \frac{1}{2}\left[\left(9 - \frac{9}{2}\right) - \left(6 - \frac{4}{2}\right)\right]$

$= \frac{1}{2}\left(\frac{1}{2}\right) + \left[\left(\frac{9}{2} - \frac{9}{4}\right) - 2\right]$

$= \frac{1}{4} + \left[\frac{9}{4} - 2\right] = 1/2$

Assume $p = 1/2$ $q = 1/2$ $n = 3$

P[exactly one value greater than 1.5]

$= 3C_1 (1/2)^1 (1/2)$

$= 3(1/2)(1/4)$

$= 3/8$

4. A Continuous random variable X has the pdf $f(x) = \begin{cases} \frac{k}{1+x^2}, & -\infty < x < \infty \\ 0, & \text{Otherwise} \end{cases}$

i) find k (ii) Distribution function of X
iii) $P[X > 0]$

Soln

i)
$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$K \int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = 1$$

$$k\left[\tan^{-1} x\right]_{-\infty}^{\infty} = 1$$

$$K\left[\tan^{-1}\infty - \tan^{-1}(-\infty)\right] = 1$$

$$K\left[\pi/2 + \pi/2\right] = 1$$

$$k \cdot \pi = 1$$

$$k = 1/\pi$$

$$\therefore f(x) = \begin{cases} \frac{1}{\pi} \cdot \frac{1}{1+x^2}, & -\infty < x < \infty \\ 0, & \text{Otherwise} \end{cases}$$

ii) In $-\infty < x < \infty$

$$F[x] = \int_{-\infty}^{x} f(x)\, dx = \int_{-\infty}^{x} \frac{1}{\pi} \frac{1}{1+x^2}\, dx$$

$$= \frac{1}{\pi}\left[\tan^{-1} x\right]_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left[\tan^{-1}x + \tan^{-1}\infty\right]$$

$$F[x] = \frac{1}{\pi}\left[\tan^{-1}x + \pi/2\right]$$

iii $P[X>0] = 1 - P[X \le 0]$

$= 1 - F[0]$

$= 1 - \frac{1}{2} = \frac{1}{2}$

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5) The Probability density function of a R.V X is given by
$$f(x) = \begin{cases} x, & 0 < x < 1 \\ k(2-x), & 1 \le x < 2 \\ 0, & \text{Otherwise} \end{cases}$$

i) Find the value of k

ii) Find $P(0.2 < x < 1.2)$

iii) What is $P(0.5 < X < 1.5 / X \ge 1)$

iv) Find the distribution function of X.

Soln

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ k(2-x), & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

i) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 x\,dx + \int_1^2 k(2-x)\,dx = 1$$

$$\left[x^2\right]^1 + k\left[2x - \frac{x^2}{2}\right]_1^2 = 1$$

$$\left(\frac{1}{2}\right) + k\left[(4-2) - (2-\tfrac{1}{2})\right] = 1$$

$$\frac{1}{2} + k\left[2 - \tfrac{3}{2}\right] = 1$$

$$\tfrac{1}{2} + \tfrac{k}{2} = 1$$

$$\tfrac{k}{2} = 1 - \tfrac{1}{2}$$

$$\boxed{k = 1}$$

$$\therefore f(x) = \begin{cases} x & , 0 < x < 1 \\ 2-x & , 1 \le x \le 2 \\ 0 & , \text{Otherwise} \end{cases}$$

(iv) Distribution function.

Case (i) $x \le 0$

$$F[x] = \int_{-\infty}^{x} f(x)\, dx = 0$$

Case (ii) $0 < x < 1$

$$F[x] = \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{x} f(x)\, dx$$

$$= 0 + \int_{0}^{x} x\, dx = \left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{2}$$

Case (iii) $1 < x < 2$

$$F[x] = \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{1} f(x)\, dx + \int_{1}^{x} f(x)\, dx$$

$$= 0 + \int_{0}^{1} x\, dx + \int_{1}^{x} (2-x)\, dx$$

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$$= \frac{1}{2} + \left[\left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2} + 2x - \frac{x^2}{2} - \frac{3}{2}$$

$$= 2x - \frac{x^2}{2} - 1$$

Case (iv) $x \geq 2$

$$F[x] = \int_{-\infty}^{x} f(x)\, dx = 1$$

(ii) $P(0.2 < X < 1.2) = F(1.2) - F(0.2)$

$$= \left[2(1.2) - \frac{(1.2)^2}{2} - 1\right] - \left[\cancel{2(0.2)} \frac{(0.2)^2}{2}\right]$$

$$= 0.68 - 0.02$$

$$= 0.66$$

(iii) $P(0.5 < X < 1.5 / X \geq 1) = \dfrac{P(0.5 < X < 1.5) \cap P(X \geq 1)}{P(X \geq 1)}$

$$= \frac{P(0.5 < X < 1.5) \cap (1 \leq x \leq 2)}{P(X \geq 1)}$$

$$= \frac{P(1 \leq X \leq 1.5)}{P(X \geq 1)}$$

$$= \frac{F(1.5) - F(1)}{1 - P(X < 1)} =$$

Mathematical Expectation

Let X be a r.v then the Mathematical expectation of X is given by

$$E[X] = \begin{cases} \sum x\, p(x), & X \text{ is discrete} \\ \int x\, f(x)\, dx, & X \text{ is continuous.} \end{cases}$$

Moments about origin

The r^{th} Moment about origin is

$$\mu_r' = E[X^r] = \begin{cases} \sum x^r p(x), & X \text{ is discrete} \\ \int x^r f(x)\, dx, & X \text{ is continuous} \end{cases}$$

$$\text{Mean} = E[X]$$

$$\text{Variance} = E[X^2] - \{E[X]\}^2$$

Moments about Mean [central Moments]

$$\mu_r = E[(X-\bar{X})^r] = \begin{cases} \sum (x-\bar{x})^r p(x), & X \text{ is discrete} \\ \int (x-\bar{x})^r f(x)\, dx, & X \text{ is Continuous} \end{cases}$$

Discrete R.V	Continuous R.V
① $E[X] = \sum x\, p(x)$	$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$
② $E[X^r] = \mu_r' = \sum x^r p(x)$	$E[X^r] = \mu_r' = \int_{-\infty}^{\infty} x^r f(x)\, dx$
③ Mean $= \mu_1' = \sum x p(x)$	Mean $\mu_1' = \int_{-\infty}^{\infty} x f(x)\, dx$
④ $\mu_2' = \sum x^2 p(x)$	$\mu_2' = \int_{-\infty}^{\infty} x^2 f(x)\, dx$
⑤ Variance $= \mu_2' - {\mu_1'}^2$ $= E[X^2] - \{E[X]\}^2$	Variance $= \mu_2' - {\mu_1'}^2$ $= E[X^2] - \{E[X]\}^2$

Note:

1. $E[aX+b] = aE[X] + b$
2. $Var(aX+b) = a^2 \, var\, X$
3. $cov(X,Y) = E[XY] - E[X].E[Y]$
4. If X & Y are independent, then $cov(X,Y) = 0$
5. $cov(aX, bY) = ab\, cov(X,Y)$
6. $cov(X+a, Y+b) = cov(X,Y)$
7. $cov(aX+b, cY+d) = ac\, cov(X,Y)$
8. $Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2\, cov(X_1, X_2)$
9. $Var(X_1 - X_2) = Var(X_1) + Var\, X_2 - 2\, Cov(X_1, X_2)$

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Note:

1) $E[X+Y] = E[X] + E[Y]$

2) $E[XY] = E[X] \cdot E[Y]$.

(1) Given the following Probability distribution of X Compute (i) $E[X]$ (ii) $E[X^2]$ (iii) $E[2X \pm 3]$ (iv) $Var(2X \pm 3)$.

X	-3	-2	-1	0	1	2	3
P(x)	0.05	0.10	0.30	0	0.30	0.15	0.10

Soln

(i) $E[X] = \sum_{i=1}^{7} x_i P(x_i)$

$= (-3)(0.05) + (-2)(0.1) + (-1)(0.30) + 0 + 1(0.30) + 2(0.15) + 3(0.10)$

$= 0.25$

(ii) $E[X^2] = \sum_{i=1}^{7} x_i^2 P(x_i)$

$= (-3)^2(0.05) + (-2)^2(0.10) + (-1)^2(0.30) + 0 + 1^2(0.30) + 2^2(0.15) + 3^2(0.10)$

$= 2.95$

(iii) $E[2X \pm 3] = 2E[X] \pm 3$

$= 2(0.25) \pm 3$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{-\infty}^{0} x^2 f(x)\, dx + \int_0^1 x^2 . x\, dx + \int_1^2 x^2 (2-x) dx + \int_2^{\infty} x^2 f(x) dx$$

$$= 0 + \int_0^1 x^3\, dx + \int_1^2 (2x^2 - x^3)\, dx + 0$$

$$= \left[\frac{x^4}{4}\right]_0^1 + \left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_1^2$$

$$= \left[\frac{1}{4} - 0\right] + \left[\left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right)\right]$$

$$= \frac{1}{4} + \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4}$$

$$= \frac{7}{6}$$

$$E[X^2] = 7/6$$

$$Var[X] = E[X^2] - [E[X]]^2$$

$$= \frac{7}{6} - 1 = \frac{1}{6}$$

<u>Moment generating function</u>:

$$M_X(t) = E[e^{tx}] = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f(x)\, dx, & X \text{ is continuous} \\ \sum_{x=-\infty}^{\infty} e^{tx} p(x), & X \text{ is discrete.} \end{cases}$$

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① Prove that r^{th} moment of the R.V 'X' about origin is $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$.

Soln

$$M_X(t) = E[e^{tx}]$$

$$= E\left[1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \dots + \frac{(tX)^r}{r!} + \dots\right]$$

$$= 1 + t\frac{E[X]}{1!} + t^2\frac{E[X^2]}{2!} + \dots + \frac{t^r}{r!}E[X^r] + \dots$$

$$= 1 + t\mu_1' + t^2\mu_2' + \dots + \frac{t^r}{r!}\mu_r' + \dots$$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

Note:

r^{th} Moment = co-efficient of $\frac{t^r}{r!}$

② Find μ_1' and μ_2' from $M_X(t)$.

Soln

WKT $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$

$$M_X(t) = \mu_0' + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{r!}\mu_r' + \dots$$

diff w.r.t 't'

$$M_X'(t) = \mu_1' + \frac{2t}{2!}\mu_2' + \dots$$

$$M_X'(0) = \mu_1' = \text{Mean}$$

$$\therefore \text{Mean} = \mu_1' = M_X'(0) = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

|||ly

$$M_X''(t) = \mu_2' + t\mu_3' + \cdots\cdots$$

$$\mu_2' = M_X''(0) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}.$$

In General

$$\mu_r' = \left[\frac{d^r}{dt^r} M_X(t)\right]_{t=0}.$$

③ Obtain the Mgf of X about the pt $X = a$.

$$M_X(t) = E\left[e^{t(X-a)}\right]$$

$$= E\left[1 + \frac{t(X-a)}{1!} + \frac{t^2}{2!}(X-a)^2 + \cdots + \frac{t^r}{r!}(X-a)^r + \cdots\right]$$

$$= 1 + tE[X-a] + \frac{t^2}{2!}E(X-a)^2 + \cdots + \frac{t^r}{r!}E(X-a)^r + \cdots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

$$\left[M_1(t)\right] = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{}$$

(4) Find the MGF of the random Variable with the probability law $P(X=x) = q^{x-1} \cdot p \quad x = 1, 2, 3, \ldots\ldots$ find the Mean and Variance.

Soln:

$$M_X(t) = E[e^{tx}]$$

$$= \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} \cdot p$$

$$= \sum_{x=1}^{\infty} e^{tx} \cdot q^{x} \cdot \frac{p}{q}$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (qe^t)^x$$

$$= \frac{p}{q}(qe^t) \sum_{x=1}^{\infty} (qe^t)^{x-1}$$

$$= pe^t \left[1 + qe^t + (qe^t)^2 + \cdots\right]$$

$$= pe^t \left[1 - qe^t\right]^{-1}$$

M.g.F is $M_X(t) = \dfrac{pe^t}{1-qe^t}$

diff w.r.t 't'

$$\frac{d}{dt} M_X(t) = \frac{(1-qe^t)\,pe^t - pe^t(-qe^t)}{\ldots}$$

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$$= \frac{pe^t - pqe^{2t} + pqe^{2t}}{(1-qe^t)^2}$$

$$M_x'(t) = \frac{pe^t}{(1-qe^t)^2} \quad \text{---} \quad (1)$$

To find Mean:

$$\mu_1' \text{ (about origin)} = M_x'(0)$$

$$= \frac{p}{(1-q)^2}$$

$$= \frac{p}{p^2} = \frac{1}{p}$$

$$\mu_1' = \text{Mean} = \frac{1}{p}$$

diff (1) w.r.t 't'

$$M_x''(t) = \frac{(1-qe^t)^2 pe^t - pe^t \, 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}$$

$$= \frac{(1-qe^t)\left[(1-qe^t)pe^t + 2pqe^{2t}\right]}{(1-qe^t)^4}$$

$$= \frac{pe^t - pqe^{2t} + 2pq\,e^{2t}}{(1-qe^t)^3}$$

$$M_x''(t) = \frac{pe^t(1+qe^t)}{(1-qe^t)^3}.$$

μ_2' (about origin) $= M_x''(0)$

$$\mu_2' = \frac{p(1+q)}{(1-q)^3} = \frac{p(1+q)}{p^3} = \frac{1+q}{p^2}$$

$$\boxed{\text{Mean } \mu_1' = \frac{1}{p}}$$

$$\text{Variance} = \mu_2' - \mu_1'^2 = \frac{1+q}{p^2} - \frac{1}{p^2}$$

$$= \frac{1+q-1}{p^2}$$

$$\boxed{\text{Variance} = \frac{q}{p^2}}$$

5) find the MGF of the random Variable X having the probability density function $f(x) = \begin{cases} \frac{x}{4}e^{-x/2}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

Also deduce the first 4 Moments about the origin.

Soln

$$f(x) = \begin{cases} \frac{x}{4} e^{-x/2}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

MGF

$$M_X(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$u = x \quad v = e^{-(\frac{1}{2}-t)x}$

$$= \int_0^{\infty} e^{tx} \cdot \frac{x}{4} e^{-x/2}\, dx$$

$$= \frac{1}{4} \int_0^{\infty} x\, e^{-(\frac{1}{2}-t)}\, dt$$

$$= \frac{1}{4} \left[\frac{x e^{-(\frac{1}{2}-t)x}}{-(\frac{1}{2}-t)} - \frac{e^{-(\frac{1}{2}-t)x}}{-(\frac{1}{2}-t)^2} \right]_0^{\infty}$$

$$= \frac{1}{4} \left[\frac{-2x e^{-(\frac{1-2t}{2})}}{(1-2t)} + \frac{4e^{(\frac{1-2t}{t})x}}{(1-2t)^2} \right]_0^{\infty}$$

$$= \frac{1}{4} \left[(0+0) - \left(0 + \frac{4e^0}{(1-2t)^2}\right) \right]$$

$$M_X(t) = \frac{1}{(1-2t)^2} = (1-2t)^{-2}$$

$$M_X'(t) = (-2)(1-2t)^{-3}(-2)$$
$$= 4(1-2t)^{-3}$$

$$M_X'(0) = 4$$

$$M_X''(t) = 4(-3)(1-2t)^{-4}(-2)$$
$$= 24(1-2t)^{-4}$$

$$M_X''(0) = 24$$

$$M_X'''(t) = 24(-4)(1-2t)^{-5}(-2)$$
$$= 192(1-2t)^{-5}$$

$$M_X'''(0) = 192$$

$$M_X^{IV}(t) = 192(-5)(1-2t)^{-6}(-2)$$
$$= 1920(1-2t)^{-6}$$

$$M_X^{IV}(0) = 1920.$$

6) Let X be a random variable with pdf

$$f(x) = \begin{cases} \frac{1}{3}e^{-x/3}, & x>0 \\ 0, & \text{otherwise} \end{cases}$$

find
a) $P(X>3)$
b) MGF of 'X'
c) $E[X]$ & var X.

Soln

$$f(x) = \begin{cases} \frac{1}{3}e^{-x/3}, & x>0 \end{cases}$$

$M_x'(t) = -(1-3t)^{-2}(-3) = 3(1-3t)^{-2}$

$E[X] = \text{Mean} = M_x'(0) = 3$

$M_x''(t) = -6(1-3t)^{-3} = 18(1-3t)^{-3}$

$M_x''(0) = 18$

$E[X^2] = 18$

$\text{Var } X = E[X^2] - [E[X]]^2$

$= 18 - 3^2$

$\text{Var } X = 9.$

7) A continous random variable X has the pdf $f(x) = kx^2e^{-x}$, $x \geq 0$ find the r^{th} moment of X about the origin. Hence find the variance of X.

Soln

To find k

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{\infty} kx^2e^{-x}\,dx = 1$$

$k\left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^{\infty} = 1$ $\quad u = x^2 \quad v = e^{-x}$; $u' = 2x \quad v_1 = -e^{-x}$; $u'' = 2 \quad v_2 = e^{-x}$; $u''' = 0 \quad v_3 = -e^{-x}$

$k\left[(\cancel{\infty} - \cancel{\infty} - 2e^{-\infty}) - (0 - 0 - 2e^{-0})\right] = 1$

$2k = 1$

$k = 1/2$

$f(x) = \frac{1}{2}x^2e^{-x}.$

find r^{th} moment:

$\mu_r' = E[x^r] = \int_{-\infty}^{\infty} x^r f(x)\,dx$

$= \int_0^{\infty} x^r \frac{1}{2} x^2 e^{-x}\,dx$

$= \frac{1}{2}\int_0^{\infty} x^{r+2}e^{-x}\,dx$ $\quad u = x^{r+2} \quad v = e^{-x}$; $u' = (r+2)x^{r+1} \quad v_1 = -e^{-x}$; $u'' = (r+2)(r+1)x^r \quad v_2 = e^{-x}$; $v_3 = -e^{-x}$

$= \frac{1}{2}\left[-e^{-x}x^{r+2} - (r+2)x^{r+1}e^{-x} - (r+2)(r+1)x^r e^{-x} - \dots - (r+2)!e^{-x}\right]_0^{\infty}$

$= -\frac{1}{2}\left[e^{-x}\left(x^{r+2} + (r+2)x^{r+1} + \dots (r+2)!\right)\right]_0^{\infty}$

$\mu_r' = -\frac{1}{2}\left(-(r+2)!\right) = \frac{1}{2}(r+2)!$

$\mu_1' = \frac{3!}{2} = 3.$ $\qquad var(x) = E[x^2] - [E[x]]^2$

$= 12 - 3^2$

$\mu_2' = \frac{4!}{2} = 12$ $\qquad = 3$

8) If the pdf of 'X' is given by

$$f(x) = \begin{cases} 2(1-x), & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$

a) Show that $E[X^r] = \frac{2}{(r+1)(r+2)}$

b) Using this result, evaluate $E[(2X+1)^2]$

Soln

$$f(x) = 2(1-x)$$

$$E[X^r] = \int_{-\infty}^{\infty} x^r f(x)\, dx$$

$$= \int_0^1 x^r 2(1-x)\, dx$$

$$= 2\int_0^1 (x^r - x^{r+1})\, dx$$

$$= 2\left[\frac{x^{r+1}}{r+1} - \frac{x^{r+2}}{r+2}\right]_0^1$$

$$= 2\left[\left(\frac{1}{r+1} - \frac{1}{r+2}\right) - (0-0)\right]$$

$$= 2\left[\frac{1}{r+1} - \frac{1}{r+2}\right]$$

$$= 2\left[\frac{(r+2)-(r+1)}{(r+1)(r+2)}\right]$$

$$= 2\left[\frac{\cancel{r}+2-\cancel{r}-1}{(r+1)(r+2}\right]$$

$$E[X^r] = \frac{2}{(r+1)(r+2)}$$

Put $r = 1$

$$E[X] = \frac{2}{(1+1)(1+2)} = \frac{1}{3}$$

$$E[X^2] = \frac{2}{(2+1)(2+2)} = \frac{1}{6}$$

$$E[(2x+1)^2] = E[4x^2+1+4x]$$
$$= E[4x^2] + E[1] + E[4x]$$
$$= 4E[x^2] + 4E[x] + 1$$
$$= \frac{4}{6} + \frac{4}{3} + 1$$
$$= \frac{4+8+6}{6}$$
$$= \frac{18}{6}$$

$E[(2x+1)^2]$

9\. Consider a discrete r.v 'x' with probability function $p(X=x) = \begin{cases} \frac{1}{x(x+1)}, & x = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$

Show that $E[x]$ does not exist even though MGF exist.

Soln

$$p(x) = \frac{1}{x(x+1)}$$

Gn. $E[x] = \sum_{x=1}^{\infty} x\, p(x)$

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$$= \sum_{x=1}^{\infty} \cancel{x} \frac{1}{\cancel{x}(x+1)}$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$= -1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$= -1 + \sum_{x=1}^{\infty} \frac{1}{x}$$

$\sum_{x=1}^{\infty} \frac{1}{x}$ is a divergent series.

$\therefore$ $E[X]$ does not exist. and hence no moment exists.

Now, MGF of X is gn by

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} p(x) = \sum_{x=1}^{\infty} \frac{e^{tx}}{x(x+1)}$$

Put $z = e^t$

$$= \sum_{x=1}^{\infty} \frac{z^x}{x(x+1)}$$

$$= \frac{z}{1\cdot 2} + \frac{z^2}{2\cdot 3} + \frac{z^3}{3\cdot 4} + \cdots$$

$$= z\left(1 - \frac{1}{2}\right) + z^2\left(\frac{1}{2} - \frac{1}{3}\right) + z^3\left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

$$= \left[z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right] - \frac{z}{2} - \frac{z^2}{3} - \frac{z^3}{4} - \cdots$$

$$= -\log(1-z) - \frac{1}{z}\left[\frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots\right]$$

$$= -\log(1-z) - \frac{1}{z}\left[-z + z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right]$$

$$= -\log(1-z) + 1 - \frac{1}{z}\left[z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right]$$

$$= -\log(1-z) + 1 - \frac{1}{z}\left[-\log(1-z)\right]$$

$= 1 + \left(\frac{1}{z} - 1\right) \log(1-z)$

$M_X(t) = 1 + (e^{-t} - 1)\log(1-e^t), \quad t < 0$

$M_X(t) = 1, \quad$ for $t = 0$.

$M_X(t)$ does not exist for $t > 0$

10) A random variable X has pdf
$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0, \\ 0, & \text{otherwise} \end{cases}$ find the MGF
when $t < 2$. find the first 4 Moments about the origin.

Soln

MGF of X is
$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$

$= \int_{-\infty}^{\infty} 2e^{-2t} e^{tx}\,dx = 2\int^{\infty} e^{-(2-t)x}\,dx$

$= 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^{\infty} = \frac{-2}{2-t}\left[e^{-\infty} - e^0\right]$

$M_X(t) = \frac{2}{2-t}$

Gn $t < 2$

$\frac{t}{2} < 1$

$\left|\frac{t}{2}\right| < 1$

$M_X(t) = \frac{2}{2(1-\frac{t}{2})} = \left(1 - \frac{t}{2}\right)^{-1}$

$= 1 + \left(\frac{t}{2}\right) + \left(\frac{t}{2}\right)^2 + \cdots \qquad \because \left|\frac{t}{2}\right| < 1$

$= 1 + \frac{1}{2}\frac{t}{1!} + \frac{2!}{4}\cdot\frac{t^2}{2!} + \frac{3!}{8}\frac{t^3}{3!} + \frac{4!}{16}\frac{t^4}{4!} + \cdots$

$\mu_1' = \frac{1}{2}$

$\mu_2' = \frac{2!}{4} = \frac{1}{2}$

$\mu_3' = \frac{3!}{8} = \frac{3}{4}$

$\mu_4' = \frac{4!}{2^4} = \frac{24}{16} = \frac{3}{2}$

Note :

If the MGF of X is

$M_X(t) = 1 + m_1 \frac{t}{1!} + m_2 \frac{t^2}{2!} + m_3 \frac{t^3}{3!} + \cdots$

μ_1' = Co-efficient of $\frac{t}{1!}$

& μ_2' = Co-efficiet of $\frac{t^2}{2!}$

$\vdots$

$= 0.608$

(iv) $P[\text{B/w } 1 \text{ and } 3 \text{ defectives}] = P[1 \le X \le 3]$

$= P[X=1] + P[X=2] + P[X=3]$

$= \left[\cancel{\left(20C_0 (1/10)^0 (9/10)^{20}\right)} + \left(20C_1 (1/10)^1 (9/10)^{19}\right) + \left(20C_2 (1/10)^2 (9/10)^{18}\right) + \left(20C_3 (1/10)^3 (9/10)^{17}\right) \right]$

$= [0.27 + 0.28517 + 0.1901178]$

$= 0.7452$

Poisson distribution:

A random variable 'x' is said to follow Poisson distribution, if its probability mass function is

$$P[X=x] = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, 3 \ldots$$

Moment Generating Function:

$M_x(t) = E[e^{tx}]$

$= \sum_{x=0}^{\infty} e^{tx} p(x)$

$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{}$

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$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{e^{tx} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} \left[\frac{(\lambda e^t)^0}{0!} + \frac{(\lambda e^t)^1}{1!} + \frac{(\lambda e^t)^2}{2!} + \dots \right]$$

$$= e^{-\lambda} \left[1 + \frac{(\lambda e^t)}{1!} + \frac{(\lambda e^t)^2}{2!} + \dots \right]$$

$$= e^{-\lambda} \left[e^{\lambda e^t} \right]$$

$$= e^{\lambda[e^t - 1]}$$

$$\boxed{\therefore M_x(t) = e^{\lambda(e^t - 1)}}$$

Mean:

$$M_x(t) = e^{\lambda(e^t - 1)}$$

$$M_x'(t) = e^{\lambda(e^t - 1)} \cdot \lambda e^t$$

$$M_x'(0) = e^{\lambda(e^0 - 1)} \cdot \lambda e^0$$

$$= e^0 \cdot \lambda e^0$$

$$= \lambda$$

$$\boxed{\therefore \text{Mean} = \lambda} = E[x]$$

To find $E[x^2]$:

$$M_x''(t) = \lambda\left[e^{\lambda(e^t-1)} \cdot e^t + e^t \cdot e^{\lambda(e^t-1)} \cdot \lambda e^t\right]$$

$$M_x''(0) = \lambda\left[e^{\lambda(e^0-1)} \cdot e^0 + e^0 \cdot e^{\lambda(e^0-1)} \cdot \lambda e^0\right]$$

$$= \lambda\left[e^0 \cdot e^0 + e^0 \cdot e^0 \cdot \lambda e^0\right]$$

$$= \lambda[1+\lambda]$$

$$= \lambda + \lambda^2$$

$$E[x^2] = \lambda + \lambda^2$$

Variance:

$$Var[x] = E[x^2] - [E[x]]^2$$

$$= \lambda + \cancel{\lambda^2} - \cancel{\lambda^2}$$

$$= \lambda$$

$$\boxed{\therefore Var[x] = \lambda}$$

Note:

In poisson distribution,

Mean = Variance = λ

27 1. The atoms of a radioactive elements are disintegrating. If every gram of this elements, an average emits 3.9 alphaparticles per second. What is the probability that during the next second the number of alpha particles emitted from 1 gram is

(i) atmost 6

(ii) atleast 2

(iii) atleast 3 and atmost 6

Solution:

Given $\lambda = 3.9$

$$P[X=x] = \frac{e^{-\lambda} \lambda^{x}}{x!}$$

(i) $P[\text{atmost } 6] = P[X \le 6]$

$$= P[X=0] + P[X=1] + P[X=2] + P[X=3] + P[X=4] + P[X=5] + P[X=6]$$

$$= e^{-3.9}\left[\frac{(3.9)^0}{0!} + \frac{(3.9)^1}{1!} + \frac{(3.9)^2}{2!} + \frac{(3.9)^3}{3!} + \frac{(3.9)^4}{4!} + \frac{(3.9)^5}{5!} + \frac{(3.9)^6}{6!}\right]$$

$= 0.0202 \, [$

$= 0.0202[44.4365]$

$= 0.8976$

(ii) $p(\text{atleast } 2) = P[x \geq 2]$

$= 1 - P[x < 2]$

$= 1 - P[x=0] + P[x=1]$

$= 1 - e^{-3.9}[1 + 3.9]$

$= 1 - 0.0202[4.9]$

$= 1 - 0.09898$

$= 0.90102$

(iii) $P[\text{atleast } 3 \text{ and atmost } 6] = P[3 \leq x \leq 6]$

$= P[x=3] + P[x=4] + P[x=5] + P[x=6]$

$= 0.0202[9.8865 + 9.6393 + 7.5186 + 4.8871]$

$= 0.0202[31.9315]$

$= 0.6450.$

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28. 2. Suppose that the number of calls coming into telephone exchange b/w 9.am and 10 am is a poisson random variable with parameter 2, and the number of telephone calls coming b/w 10 AM and 11 AM is a random variable with parameter 6. If these two random variables are independent. What is the probability that more than 5 calls come in between 9 AM and 11 AM.

Solution!

Let X_1 – calls b/w 9.AM and 10 AM with $\lambda_1 = 2$

X_2 – Calls b/w 10 AM and 11 A.M with $\lambda_2 = 6$

WKT,

$$X = X_1 + X_2.$$

$$\lambda = \lambda_1 + \lambda_2$$

$$\therefore \lambda = 2 + 6 = 8$$

$$\boxed{\lambda = 8}$$

X – calls b/w 9 AM and 11 AM with $\lambda = 8$

$$P[X = x] = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$P[X > 5] = 1 - P[X \le 5]$$

$$= 1 - \Big[P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) \Big]$$

$$= 1 - e^{-8}\left[\frac{8^0}{0!} + \frac{8^1}{1!} + \frac{8^2}{2!} + \frac{8^3}{3!} + \frac{8^4}{4!} + \frac{8^5}{5!}\right]$$

$$= 1 - 3.354 \times 10^{-4}\left[1 + 8 + 32 + 85.33 + 170.666 + 273.06\right]$$

$$= 1 - 3.354 \times 10^{-4}\left[570.056\right]$$

$$= 1 - 0.1911$$

$$= 0.808$$

29. 3. The MGF of a random variable X be $e^{4(e^t - 1)}$. Show that $P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.93$

Solution:

Given,

$$MGF = e^{4(e^t-1)} = M_x(t) \quad \text{---} (1)$$

In poisson distribution.

$$MGF = M_x(t) = e^{\lambda(e^t - 1)} \quad \text{---} (2)$$

On comparing (1) & (2)

$$\lambda = 4$$

In poisson distribution

Mean = Variance = λ

$$\mu = \sigma^2 = 4$$

To prove:

$P[\mu - 2\sigma < X < \mu + 2\sigma] = 0.93$

LHS

$P[4 - 2(2) < X < 4 + 2(2)] = P[0 < X < 8)$

$= P[X=1] + P[X=2] + P[X=3] + P[X=4] +$

$P[X=5] + P[X=6] + P[X=7)$

$= e^{-4}\left[\frac{4^1}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!} + \frac{4^6}{6!} + \frac{4^7}{7!}\right)$

$= e^{-4}$

$= 0.01831[4 + 8 + 10.66 + 10.666 + 8.533 + 5.6888 + 3.2507]$

$= 0.01831[50.8045)$

$= 0.93$

---

30. A. If X is a poisson random Variable such that $P[X=2] = 9P[X=4] + 90P[X=6]$. Find the Variate.

Solution:

$P[X=x] = \frac{e^{-\lambda}\lambda^x}{x!}$

$P[X=2] = 9P[X=4] + 90P[X=6]$

$\frac{e^{-\lambda}\lambda^2}{} \quad \frac{e^{-\lambda}\lambda^4}{} \quad 90\frac{e^{-\lambda}\lambda^6}{}$

$$\frac{1}{1\times \cancel{2}} = \frac{\cancel{9}^{3}\lambda^2}{\cancel{1\times 2}\times 3\times 4} + \frac{\cancel{90}^{30}\lambda^4}{1\times 2\times 3\times 4\times 5\times 6}$$

$$1 = \frac{3\lambda^2}{4} + \frac{\lambda^4}{4}$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

$$(\lambda^2)^2 + 3(\lambda^2) - 4 = 0$$

$$(\lambda^2 - 1)(\lambda^2 + 4) = 0$$

$\lambda^2 = 1 \qquad \lambda^2 + 4 = 0$

$\lambda = 1 \qquad \lambda^2 = -4$

$\lambda^2 = -4$ not possible.

$\therefore \lambda = 1$

Variance $= \lambda = 1$

23.1.13

Geometric distribution

A random variable 'x' is said to follow geometric distribution if its probability mass function is

$$P[X=x] = q^{x-1}p, \quad x = 1, 2$$

MGF:

$$M_X(t) = E[e^{tx}]$$

$$= \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} \cdot p$$

$$= \sum_{x=1}^{\infty} e^{tx} \cdot q^{x} \cdot q^{-1} \cdot p$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (qe^{t})^{x}$$

$$= \frac{p}{q} \left[(qe^{t}) + (qe^{t})^{2} + (qe^{t})^{3} + \dots \right]$$

$$= \frac{p}{q} \cdot qe^{t} \left[1 + (qe^{t}) + (qe^{t})^{2} + \dots \right]$$

$$= pe^{t} \left[1 - qe^{t} \right]^{-1}$$

$$\boxed{M_X(t) = \frac{pe^{t}}{1-qe^{t}}}$$

Mean:

$$M_x(t) = \frac{pe^t}{1-qe^t}$$

$$M_x'(t) = \frac{(1-qe^t)pe^t - pe^t(-qe^t)}{(1-qe^t)^2}$$

$$= \frac{pe^t - \cancel{pq}e^{2t} + p\cancel{q}e^{2t}}{(1-qe^t)^2}$$

$$\cancel{M_x'(0) = pe^0 - pqe^0}$$

$$M_x'(t) = \frac{pe^t}{(1-qe^t)^2}$$

$$M_x'(0) = \frac{pe^0}{(1-qe^0)^2}$$

$$= \frac{p}{(1-q)^2}$$

$$= \frac{p}{p^2}$$

$$= \frac{1}{p}$$

$\therefore$ Mean $= E[x] = \frac{1}{p}$

To find : $E[x^2]$

$$M_x'(t) = \frac{pe^t}{(1-qe^t)^2}$$

$$M_x''(t) = \frac{(1-qe^t)^2 pe^t - pe^t(2)(1-qe^t)(-qe^t)}{}$$

$$= \frac{(1-qe^t)\left[(1-qe^t)pe^t + 2pqe^{2t}\right]}{(1-qe^t)^{\cancel{4}\,3}}$$

$$= \frac{pe^t - pqe^{2t} + 2pqe^{2t}}{(1-qe^t)^3}$$

$$= \frac{pe^t + pqe^{2t}}{(1-qe^t)^3}$$

$$M_x''(0) = \frac{pe^0 + pqe^0}{(1-qe^0)^3}$$

$$= \frac{p+pq}{(1-q)^3}$$

$$= \frac{p(1+q)}{p^{\cancel{3}\,2}}$$

$$= \frac{1+q}{p^2}$$

$$M_x''(0) = E[x^2] = \frac{1+q}{p^2}$$

Variance:

$$Var[x] = E[x^2] - (E[x])^2$$

$$= \frac{1+q}{p^2} - \left(\frac{1}{p}\right)^2$$

$$= \frac{\cancel{1}+q-\cancel{1}}{p^2}$$

$$= \frac{q}{p^2}$$

Memoryless property:

The memoryless property is given by $P[x > s+t / x > s] = P[x > t]$ for any $s, t > 0$.

Consider:

$$P[X > s+t] = \sum_{x=s+t+1}^{\infty} p q^{x-1}$$

$$= p\left[q^{s+t} + q^{s+t+1} + q^{s+t+2} + q^{s+t+3} + \dots\right]$$

$$= p q^{s+t}\left[1 + q + q^{2} + \dots\right]$$

$$= p q^{s+1}\left[1-q\right]^{-1}$$

$$= \frac{p q^{s+1}}{1-q}$$

$$P[x > s+t] = \frac{p q^{s+t}}{1-q}$$

$$= \frac{p q^{s+t}}{p}$$

$$= q^{s+t}$$

$$\therefore P[x > s+t] = q^{s+t} \quad \text{——} (1)$$

$$\therefore P[x > s] = q^{s} \quad \text{——} (2)$$

$$P[x > t] = q^{t} \quad \text{——} (3)$$

$$P[X>s+t/X>s] = \frac{P[X>s+t \cap X>s]}{P[X>s]}$$

$$= \frac{P[X>s+t]}{P[X>s]}$$

$$= \frac{q^{s+t}}{q^{s}}$$

$$= q^{t}$$

$$= P[X>t]$$

$$\therefore P[X>s+t/X>s] = P[X>t]$$

---

31. 1. Let one copy of a magazine out of 10 copies bears a special price following distribution determine its mean and variance.

Solution:

Given

$$P = \frac{1}{10}$$

$$\therefore P+q=1$$

$$q = 1-P$$

$$= 1-\frac{1}{10}$$

$$\therefore q = \frac{9}{10}$$

Mean:

$= 10$

$\therefore$ Mean $= 10$

Variance:

$$Var[x] = \frac{q}{p^2}$$

$$= \frac{9}{10} \Big/ \left(\frac{1}{10}\right)^2$$

$$= \frac{9 \times 10\cancel{0}}{1\cancel{0}}$$

$$= 90$$

$\therefore$ Variance $= 90$.

Uniform distribution [rectangular distribution]

A random variable x is said to have a continuous uniform distribution, if its probability density function is given by.

$$f(x) = \begin{cases} 1/b-a, & a < x < b \\ 0 & \text{Otherwise}. \end{cases}$$

MGF:

$$M_x(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

b

$$= \frac{1}{b-a} \int_a^b e^{tx} \, dx$$

$$= \frac{1}{b-a} \left[\frac{e^{tx}}{t} \right]_a^b$$

$$= \frac{1}{(b-a)t} \left[e^{bx} - e^{ax} \right]$$

$$= \frac{e^{bx} - e^{ax}}{(b-a)t}$$

$$M_X(t) = \frac{e^{bx} - e^{ax}}{(b-a)t}.$$

Mean:

$$E[X] = \int_{-\infty}^{\infty} x \, f(x) \, dx$$

$$= \int_a^b x \cdot \frac{1}{b-a} \, dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b$$

$$= \frac{1}{2(b-a)} (b^2 - a^2)$$

$$= \frac{(b+a)\cancel{(b-a)}}{2\cancel{(b-a)}}$$

$$= \frac{b+a}{2}$$

$\therefore E[X] = \text{Mean} = \frac{b+a}{}$

Variance:

To find $E[x^2]$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_a^b x^2 \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{3(b-a)}\left[b^3 - a^3\right]$$

$$= \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}}$$

$$= \frac{b^2+ab+a^2}{3}.$$

$$E[x^2] = \frac{a^2+ab+b^2}{3}$$

$$Var[x] = E[x^2] - (E[x])^2$$

$$= \frac{a^2+ab+b^2}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{a^2+ab+b^2}{3} - \frac{(a^2+b^2+2ab)}{4}$$

$$= \frac{4a^2+4ab+4b^2-3a^2-3b^2-6ab}{12}$$

$$= \frac{a^2+b^2-2ab}{12}$$

$$= \frac{(a-b)^2}{}$$

32. 1. If X is uniformly distributed over $(0,10)$ find the probability that

(i) $(X<2)$

(ii) $(X>8)$

(iii) $(3<X<9)$

Solution:

$$f(x) = \begin{cases} \frac{1}{b-a} , & a \le x < b \\ 0 , & \text{Otherwise} \end{cases}$$

$(0,10)$

$$f(x) = \begin{cases} \frac{1}{10} ; & 0 < x < 10 \\ 0 , & \text{Otherwise.} \end{cases}$$

(i) $P[X<2]$

$$f(x) = \int_0^2 f(x)\,dx$$

$$= \int_0^2 \frac{1}{10}\,dx$$

$$= \frac{1}{10}\left[x\right]_0^2$$

$$= \frac{1}{10}[2-0]$$

$$= \frac{2}{10}$$

$$= \frac{1}{5}$$

$\therefore P[X<2] = 1/5$

(ii) $P[x > 8]$

$$f(x) = \int_{8}^{10} \frac{1}{10}\, dx$$

$$= \frac{1}{10}\left[x\right]_{8}^{10}$$

$$= \frac{1}{10}[10-8]$$

$$= \frac{2}{10}$$

$$= \frac{1}{5}$$

$\therefore P[x>8] = \frac{1}{5}$

(iii) $P[3 < x < 9]$

$$f(x) = \int_{3}^{9} \frac{1}{10}\, dx$$

$$= \frac{1}{10}\left[x\right]_{3}^{9}$$

$$= \frac{(9-3)}{10}$$

$$= \frac{6}{10}$$

$$= 3/5$$

$\therefore P[x>8] = 3/5.$

33 2. If X is uniformly distributed over $(-\alpha, \alpha)$, $\alpha < 0$, find α, so ~~Show~~ that

(i) $P[X > 1] = \frac{1}{3}$

(ii) $P[|X| < 1) = P(|X| > 1])$

Solution:

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{Otherwise.} \end{cases}$$

$a = -\alpha$, $b = \alpha$.

$$f(x) = \begin{cases} \frac{1}{2\alpha}, & -\alpha < x < \alpha \\ 0, & \text{otherwise.} \end{cases}$$

(i) $P[X > 1] = \frac{1}{3}$

$$\int_1^{\alpha} \frac{1}{2\alpha}\, dx = \frac{1}{3}$$

$$\frac{1}{2\alpha}[x]_1^{\alpha} = \frac{1}{3}$$

$$\frac{1}{2\alpha}[\alpha - 1] = \frac{1}{3}$$

$$3\alpha - 3 = 2\alpha$$

$$3\alpha - 2\alpha - 3 = 0$$

$$\alpha - 3 = 0$$

$$\alpha = 3$$

$$\therefore \boxed{\alpha = 3}$$

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(ii) $P[|x|<1] = P[|x|>1]$

$$P[|x|<1] = 1 - P(|x|<1)$$

$$2P[|x|<1] = 1$$

$$2P[-1<x<1] = 1$$

$$2\int_{-1}^{1} \frac{1}{2\alpha}\,dx = 1$$

$$\frac{2}{2\alpha}\left[x\right]_{-1}^{1} = 1$$

$$\frac{1}{\alpha}\left[1-(-1)\right] = 1$$

$$\frac{1}{\alpha}\left[2\right] = 1$$

$$\boxed{\alpha = 2}$$

34\. 3. Four buses arrive at a specified stop at 15 min intervals starting at 7 AM. (ie) they arrive at 7, 7.15, 7.30, 7.45. AM and so on. If a passenger arrives at a time (ie) uniformly distributed between 7 and 7.30 AM. Find the probability that he waits

(a) Less than 5 mins for a bus.

(b) more than 10 mins for a bus.

Solution:

Let 'x' denote the number of minutes passed 7, that the passenger arrived busstop in (0, 30)

$$f(x) = \begin{cases} \frac{1}{30}, & 0 < x < 30 \\ 0, & \text{Otherwise.} \end{cases}$$

(i) A passenger will have to wait less than 5 minutes, if he arrives between 7.10 and 7.15 and if he arrives between 7.25 and 7.30.

$P(10 < X < 15) + P(25 < X < 30)$

$$= \int_{10}^{15} \frac{1}{30}\,dx + \int_{25}^{30} \frac{1}{30}\,dx$$

$$= \frac{1}{30}\left[\left[x\right]_{10}^{15} + \left[x\right]_{25}^{30}\right]$$

$$= \frac{1}{30}\left[(15-10) + (30-25)\right]$$

$$= \frac{1}{30}\left[5+5\right]$$

$$= \frac{10}{30}$$

$$= \frac{1}{3}$$

$$\therefore \left[P[10 < X < 15) + P(25 < X < 30)\right] = \frac{1}{3}$$

(ii) A passenger will have to wait more than ten minutes if he arrives b/w 7 and 7.05, (or) b/w 7.15 and 7.20.

$P[0 < X < 5] + P[15 < X < 20]$

$= \int_0^5 \frac{1}{30} dx + \int_{15}^{20} \frac{1}{30} dx$

$= \frac{1}{30} \left[[x]_0^5 + [x]_{15}^{20} \right]$

$= \frac{1}{30} \left[(5-0) + (20-15) \right]$

$= \frac{1}{30} [5+5]$

$= \frac{10}{30}$

$= \frac{1}{3}$

$P[0 < X < 5] + P[15 < X < 20] = \frac{1}{3}.$

Exponential distribution

A continuous random variable x is said to follow an exponential distribution with parameter $\lambda > 0$, if its probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} , & x > 0 \\ 0 , & \text{otherwise.} \end{cases}$$

MGF:

$$M_x(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx.$$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^{\infty} e^{tx} e^{-\lambda x}\, dx$$

$$= \lambda \int_0^{\infty} \left[e^{-(\lambda - t)x}\, dx \right]$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda - t)} \right]_0^{\infty}$$

$$= \frac{\lambda}{-(\lambda - t)} \left[e^{-\infty} - e^{0} \right]$$

$$= \frac{\lambda}{\lambda - t}$$

$$M_x(t) = \frac{\lambda}{\lambda - t}, \lambda > t$$

Mean:

$$M_x(t) = \lambda (\lambda - t)^{-1}$$

$$M_x'(t) = \lambda (-1)(\lambda - t)^{-2} \cdot (-1)$$

$$= \lambda (\lambda - t)^{-2}$$

$$= \frac{\lambda}{(\lambda - t)^2}$$

$$M_x'(0) = \frac{\lambda}{(\lambda - 0)^2}$$

$$= \frac{\lambda}{\lambda^2}$$

$$= \frac{1}{\lambda}$$

$$\boxed{\therefore \text{Mean} = E[x] = \frac{1}{\lambda}}$$

To find $E[x^2]$:

$$M_x'(t) = \lambda (\lambda - t)^{-2}$$

$$M_x''(t) = \lambda (-2)(\lambda - t)^{-3} \cdot (-1)$$

$$= \frac{2\lambda}{(\lambda - t)^3}$$

$$\text{Variance} = E[x^2] - (E[x])^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{2-1}{\lambda^2}$$

$$= 1/\lambda^2$$

$$\therefore \boxed{\text{Variance} = \frac{1}{\lambda^2}}$$

Memoryless property:

If x is exponentially distributed, then $P[x > s+t / x > s]$ $= P(x > t)$ for any $s, t > 0$.

Solution:

$$P[x > k) = \int_k^\infty f(x)\, dx.$$

$$= \int_k^\infty \lambda e^{-\lambda x} dx$$

$$= \lambda \int_k^\infty e^{-\lambda x} dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_k^\infty$$

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$$= \frac{\lambda}{-\lambda}\left[-e^{-\lambda k}\right]$$

$$= e^{-\lambda k}.$$

$$P[x>s+t/x>s) = \frac{P[(x>s+t)\cap(x>s)]}{P[x>s]}$$

$$= \frac{P[x>s+t]}{P[x>s]}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda s-\lambda t} \cdot e^{\lambda s}$$

$$= e^{-\cancel{\lambda s}-\lambda t+\cancel{\lambda s}}$$

$$= e^{-\lambda t}$$

$$= P[x>t].$$

$$\therefore P[x>s+t/x>s] = P[x>t]$$

35. i. The time (in hrs) required to repair a machine is exponentially distributed with parameter $\lambda = 1/2$

(a) what is the probability that the

Given that its duration exceeds 8 hrs.?

Solution:

Let 'x' be the random Variable which represents the time to repair the machine.

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{2} e^{-x/2}, & x > 0. \end{cases}$$

(i) $P[X > 2] = e^{-\lambda 2}$ $\quad [\because P[X > k] = e^{-\lambda k}]$

$= e^{-2/2}$

$= e^{-1}$

$= 0.3678$

(ii) $P[X \geq 11 / X > 8] = P[X > 8+3 / X > 8]$

$= P[X > 3]$ $\quad (\because P[X > s+t / X > s] = P[X > t])$

$= e^{-3\lambda}$

$= e^{-3/2}$

$= 0.2231$

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25/1/13 2. If x is a random variable which follows an exponential distribution with parameter λ with $P[x \le 1) = P[x > 1]$ Find Variance of x?

Solution:

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

$$P(x \le 1) = P(x > 1)$$

$$1 - P(x > 1) = P(x > 1)$$

$$2P(x > 1) = 1$$

$$P(x > 1) = 1/2$$

$$e^{-\lambda} = 1/2 \qquad \left[\because P[x > k] = e^{-\lambda k}\right]$$

$$\cancel{\log}\ \frac{1}{e^{\lambda}} = \frac{1}{2} \Rightarrow e^{\lambda} = 2$$

Taking log on both sides,

$$\lambda = \log_e 2$$

$$Var[x] = \frac{1}{\lambda^2}$$

$$= \frac{1}{(\log_e 2)^2}$$

$$\therefore Var[x] = \frac{1}{(\log_e 2)^2}$$

Gamma distribution:

The continuous random Variable 'x' is said to follow a Gamma distribution with parameter λ, if its probability function is given by

$$f(x) = \begin{cases} \dfrac{e^{-x} x^{\lambda-1}}{\Gamma\lambda} & \lambda>0,\ 0<x<\infty \\ 0 & ,\text{ otherwise} \end{cases}$$

MGF:

$$M_x(t) = E[e^{tx}]$$

$$= \int_0^\infty e^{tx} f(x)\, dx$$

$$= \int_0^\infty e^{tx} \frac{e^{-x} x^{\lambda-1}}{\Gamma\lambda} dx$$

$$= \frac{1}{\Gamma\lambda} \int_0^\infty e^{tx} e^{-x} x^{\lambda-1} dx$$

$$= \frac{1}{\Gamma\lambda} \int_0^\infty e^{-(1-t)x} x^{\lambda-1} dx$$

Put

$u = (1-t)x$	$x \to 0 \Rightarrow u \to 0$
$du = (1-t)\,dx$	$x \to \infty \Rightarrow u \to \infty$

$$= \frac{1}{\Gamma\lambda} \int_0^\infty e^{-u} \left(\frac{u}{1-t}\right)^{\lambda-1} \cdot \frac{du}{1-t}$$

$$= \frac{1}{\Gamma\lambda} \int_0^\infty e^{-u} \frac{u^{\lambda-1}}{(1-t)^{\lambda-1}} \cdot \frac{du}{(1-t)}$$

$$= \frac{1}{\Gamma\lambda} \cdot \frac{1}{(1-t)^\lambda} \int_0^\infty e^{-u} u^{\lambda-1} du.$$

$$= \frac{1}{\cancel{\Gamma\lambda}(1-t)^\lambda} \cdot \cancel{\Gamma\lambda} \quad \left[\because \Gamma n = \int_0^\infty e^{-x} x^{n-1} \cdot dx\right]$$

$$= \frac{1}{(1-t)^\lambda}$$

$$\therefore M_x(t) = \frac{1}{(1-t)^\lambda}$$

Mean:

$$M_x(t) = (1-t)^{-\lambda}$$

$$M_x'(t) = -\lambda(1-t)^{-\lambda-1} \cdot (-1)$$

$$= \lambda(1-t)^{-(\lambda+1)}$$

$$= \frac{\lambda}{(1-t)^{\lambda+1}}$$

$$M_x'(0) = \frac{\lambda}{(1-0)^{\lambda+1}}$$

$$= \frac{\lambda}{(1)^{\lambda+1}} = \frac{\lambda}{1} = \lambda$$

$E[x^2]$:-

$$M_x'(t) = \frac{\lambda}{(1-t)^{\lambda+1}}$$

$$= \lambda (1-t)^{-(\lambda+1)}$$

$$M_x''(t) = \lambda \cdot (-(\lambda+1)) (1-t)^{-(\lambda+2)} (-1)$$

$$= \lambda(\lambda+1)(1-t)^{-(\lambda+2)}$$

$$M_x''(0) = \lambda(\lambda+1)(1-0)^{-(\lambda+2)}$$

$$= \lambda(\lambda+1)$$

$$= \lambda^2+\lambda$$

$$E[x^2] = \lambda^2+\lambda$$

Variance:

$$Var[x] = E[x^2] - (E[x])^2$$

$$= \cancel{\lambda^2} + \lambda - \cancel{\lambda^2}$$

$$= \lambda$$

$\therefore$ $Var[x] = \lambda$

Normal distribution:

A continuous random variable X is said to follow a normal distribution with mean μ, and variance σ^2, if its density function is given by the probability law,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty,$$

$$\sigma > 0, \quad -\infty < \mu < \infty.$$

MGF:

$$M_X(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx.$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx.$$

Put $z = \dfrac{x-\mu}{\sigma}$ | $x \to -\infty \Rightarrow z \to -\infty$

$dz = \dfrac{dx}{\sigma}$ | $x \to \infty \Rightarrow z \to \infty$

$\sigma\, dz = dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z+\mu)} \cdot e^{\frac{-z^2}{2}} \cdot \sigma\, dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\sigma z} \cdot e^{-\frac{z^2}{2}} \, dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z^2 - 2t\sigma z)}{2}} \, dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2 \left(z^2 - 2t\sigma z + \cancel{(t\sigma)^2}\right) + (t\sigma)^2/2} \, dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2 (z - t\sigma)^2} \, e^{\frac{t^2\sigma^2}{2}} \, dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \, e^{\frac{t^2\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-1/2 (z - t\sigma)^2} \, dz$$

$$= \frac{e^{\mu t + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} \, du$$

$$= \frac{e^{\mu t + \frac{t^2\sigma^2}{2}}}{\cancel{\sqrt{2\pi}}} \cdot \cancel{\sqrt{2\pi}}$$

$$M_x(t) = e^{\mu t + \frac{t^2\sigma^2}{2}}$$

Mean:

$$M_x(t) = e^{\mu t + \frac{t^2\sigma^2}{2}}$$

$$M_x'(t) = e^{\mu t + \frac{t^2\sigma^2}{2}} (\mu + t\sigma^2)$$

$= \mu$

$$\boxed{\text{Mean} = E[X] = \mu.}$$

To find $E[X^2]$:

$$M_x''(t) = e^{\mu t + \frac{t^2\sigma^2}{2}} \cdot \sigma^2 + (\mu + t\sigma^2)\, e^{\mu t + \frac{t^2\sigma^2}{2}} (\mu + t\sigma^2)$$

$$M_x''(0) = e^{0} \cdot \sigma^2 + (\mu + 0)\, e^{0} (\mu + 0)$$

$$E[X^2] = \sigma^2 + \mu^2$$

Variance:

$$Var[X] = E[X^2] - (E[X])^2$$

$$= \sigma^2 + \cancel{\mu^2} - \cancel{\mu^2}$$

$$= \sigma^2$$

$$\therefore \boxed{\text{Variance} = \sigma^2}$$

Standard normal distribution.

$$Z = \frac{x - \mu}{\sigma}$$ with parameter μ and σ

Basic properties:

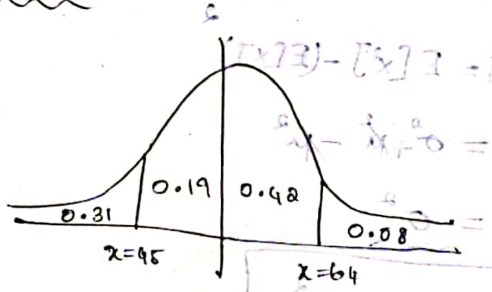
* Total area under the standard normal curve is equal to 1.

* The standard normal curve is asymptotic to x-axis.

The Standard normal curve is Symmetric about zero, most of the area under the Standard normal curve lies b/w -3 and 3.

37 1. In a normal distribution 31 % of the items are under 45 and 8 % are over 64. Find the mean and Standard deviation.

Solution:



~~The Values of z corresponding to the area 0.19 is~~

Let the mean and standard deviation of the given normal distribution be μ and σ.

The value of z corresponding to the area 0.19 is 0.5 nearly

$$\frac{45-\mu}{\sigma} = -0.5$$

$$-0.5\sigma + \mu = 45 \quad \text{——(1)}$$

The value of z corresponding to the area 0.42 is 1.4 nearly.

$$\frac{64-\mu}{\sigma} = 1.4$$

$$1.4\sigma + \mu = 64 \quad \text{———} (2)$$

Solve (1) & (2)

$$-0.5\sigma + \cancel{\mu} = 45$$
$$1.4\sigma + \cancel{\mu} = 64$$
$$(-) \quad (-) \quad (-)$$
$$-1.9\sigma = -19$$

$$\sigma = \frac{-19}{-1.9}$$

$$\sigma = 10$$

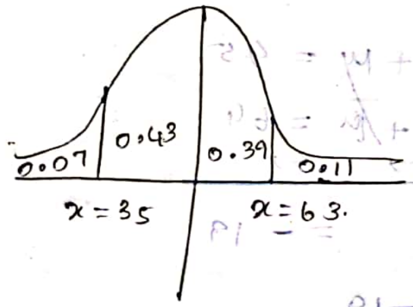
Sub $\sigma = 10$ in (2)

$$(1.4)10 + \mu = 64$$
$$14 + \mu = 64$$
$$\mu = 64 - 14$$
$$\mu = 50$$

$\therefore \sigma = 10, \mu = 50.$

2. In a distribution, exactly normal 7% of the items are under 35 and 39% items are under 63. What are the mean and Standard deviation of the distribution.

Solution :



Let the mean and Standard deviation of the normal distribution be μ and σ.

The Value of Z corresponding to the area 0.43 is 1.47 nearly.

$$\frac{35-\mu}{\sigma} = -1.47$$

$$-1.47\sigma + \mu = 35 \quad \text{---(1)}$$

The Value of Z corresponding to the area 0.39 is 1.2 nearly.

$$\frac{63-\mu}{\sigma} = 1.2$$

$$1.2\sigma + \mu = 63 \quad \text{---(2)}$$

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Solve (1) & (2)

$$-1.47\sigma + \mu = 35$$
$$1.20\sigma + \mu = 63$$
$$(-) \quad (-) \quad (-)$$

$$-1.67\sigma = 28$$

$$\sigma = \frac{28}{1.67}$$

$$\sigma = 10.48$$

$$\mu = 50.41.$$

3. The marks obtained by a number of students for a certain subject is assumed to be normally distributed with mean 65 and standard deviation 5. if 3 students are taken at random from this set, what is the probability that exactly ~~two~~ one of them will have marks over ~~70~~ 75?

Let 'x' be the random Variable which denotes the marks obtained by students.

Given:

$\mu = 65$

$\sigma = 5$

The standard normal Variation is

$$Z = \frac{x-\mu}{\sigma} = \frac{x-65}{5}$$

To find $P(x > 70)$

When $x = 70$

$$Z = \frac{70-65}{5} = \frac{5}{5} = 1$$

$$P(x > 70) = P(Z > 1)$$

$$= 0.5 - P(0 < z < 1)$$

$$= 0.5 - 0.3413$$

$$= 0.1587$$

$P(\text{a student score} > 70) = 0.1587$

$p = 0.1587$

$q = 1 - 0.1587 = 0.8413$

$n = 3$.

$P(x=x) = nCx \; p^{x} q^{n-x}$

Solution:

Let 'X' be the random Variable denoting the life time of a light bulb.

Given

$\mu = 800$

$\sigma = 40$

$Z = \frac{X - \mu}{\sigma}$

$= \frac{X - 800}{40}$

(i) P (a bulb burns more than 834 hrs)

$= P(X > 834)$

When $X = 834$ $Z = \frac{834 - 800}{40} = \frac{34}{40} = 0.85$

$P(X > 834) = P(Z > 0.85)$

$= 0.5 - P(0 < Z < 0.85)$

$= 0.5 - 0.3023$

$= 0.1977$

(ii) $P(778 < X < 834)$

When $X = 778 \Rightarrow Z = \frac{778 - 800}{40} = -0.55$

When $X = 834 \Rightarrow Z = 0.85$

$P(778 < X < 834) = P(-0.55 < Z < 0.85)$

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UNIT-II

Two DIMENSIONAL RANDOM VARIABLE

Let 'S' be the Sample Space. Let $X = X(s)$ and $Y = Y(s)$ be the two functions each assigning a real number to each outcome. $s \in S$. Then (X, Y) is a two dimensional random variable.

Note:

The two random Variables of (X, Y) are said to be independent if,

$$P[X = x_i / Y = y_j] = P[X = x_i)\ P(Y = y_j)$$

$$P_{ij} = P_{i.} \times P_{.j}$$

Problems based on marginal distribution

1. From the following joint distribution of X and Y. Find the marginal distribution.

Y \ X	0	1	2
0	3/28	9/28	3/28

The maginal distribution of X are.

$P[X=0] = P(0,0)+P(0,1)+P(0,2)$

$= \frac{3}{28}+\frac{3}{14}+\frac{1}{28}$

$= \frac{10}{28}$

$P[X=1] = P(1,0)+P(1,2)+P(1,2)$

$= \frac{9}{28}+\frac{3}{14}+0$

$= \frac{15}{28}$

$P[X=2] = P(2,0)+P(2,1)+P(2,2)$

$= \frac{3}{28}$

The marginal distribution of Y are

$P[Y=0] = P[0,0]+P[1,0]+P[2,0]$

$= \frac{3}{28}+\frac{9}{28}+\frac{3}{28}$

$= \frac{15}{28}$

$P[Y=1] = P(0,1)+P(1,1)+P(2,1)$

$= \frac{3}{14}+\frac{3}{14}$

$= \frac{6}{14}$

$P[Y=2] = P(0,2)+P(1,2)+P(2,2)$

$= \frac{1}{28}$

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The marginal distribution of X & Y are.

Y \ X	0	1	2	P(Y=y)
0	$3/28$	$9/28$	$3/28$	$15/28$
1	$3/14$	$3/14$	0	$6/14$
2	$1/28$	0	0	$1/28$
P(X=x)	$5/14$	$15/28$	$3/28$	1

1/2/13 2. If the joint p.d.f of (X,Y) is given by,

$P(x,y) = K(2x+3y)$, $x = 0,1,2$; $y = 1,2,3$. Find the marginal distribution. Also find the probability distribution of $(X+Y)$

Solution:

$P(x,y) = K(2x+3y)$

$P(0,1) = K(2(0)+3(1)) = 3K$.

$P(0,2) = K(2(0)+3(2)) = 6K$.

$P(0,3) = K(2(0)+3(3)) = 9K$.

$P(1,1) = K(2(1)+3(1)) = 5K$

$P(1,2) = K(2(1)+3(2)) = 8K$.

$P(1,3) = K(2(1)+3(3)) = 11K$

$P(2,1) = K(2(2)+3(1)) = 7K$

$P(2,2) = K(2(2)+3(2)) = 10K.$

$P(2,3) = K(2(2)+3(3)) = 13K$

To find K:

y \ x	0	1	2	P(Y=y)
1	3K	5K	7K	15K
2	6K	8K	10K	24K
3	9K	11K	13K	33K
P(X=x)	18K	24K	30K	72K

$72K = 1$

$$\boxed{K = \frac{1}{72}}$$

The marginal distribution of X & Y is

Y \ X	0	1	2	P(Y=y)
1	$\frac{3}{72}$	$\frac{5}{72}$	$\frac{7}{72}$	$\frac{15}{72}$
2	$6/72$	$8/72$	$10/72$	$24/72$
3	$9/72$	$11/72$	$13/72$	$33/72$
P(X=x)	$18/72$	$24/72$	$\frac{30}{72}$	$\frac{72}{72}=1$

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Probability distribution of X+Y

X+Y	Probability
1	$P(0,1) = 3/72$
2	$P(0,2)+P(1,1) = 6/72 + 5/72 = \frac{11}{72}$
3	$P(2,1)+P(1,2)+P(0,3) = \frac{7}{72}+\frac{8}{72}+\frac{9}{72} = \frac{24}{72}$
4	$P(2,2)+P(1,3) = \frac{10}{72}+\frac{11}{72} = \frac{21}{72}$
5	$P(2,3) = \frac{13}{72}$

Problems based on Conditional distribution

3. From the following table for by Variate distribution (X,Y). Find

(i) $P(X \le 1)$

(ii) $P(Y \le 3)$

(iii) $P(X \le 1, Y \le 3)$

(iv) $P(X \le 1 / Y \le 3)$

(v) $P(Y \le 3 / X \le 1)$

(vi) $P(X+Y \le 4)$

X \ Y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$1/8$
2	$1/32$	$1/32$	$1/64$	$1/64$	0	$2/64$

Solution:

X \ Y	6 $P(0,6)$	1 $P(0,1)$	2 $P(0,2)$	3 $P(0,3)$	4 $P(0,4)$	5 $P(0,5)$	$P(X=x)$
0	$3/32$ $P(1,6)$	0 $P(1,1)$	0 $P(1,2)$	$1/32$ $P(1,3)$	$2/32$ $P(1,4)$	$2/32$ $P(1,5)$	$8/32$ $P(X=0)$
1	$1/8$ $P(2,6)$	$1/16$ $P(2,1)$	$1/16$ $P(2,2)$	$1/8$ $P(2,3)$	$1/8$ $P(2,4)$	$1/8$ $P(2,5)$	$10/16$ $P(X=1)$
2	$2/64$	$1/32$	$1/32$	$1/64$	$1/64$	0	$8/64$ $P(X=2)$
$P(Y=y)$	$P(Y=0)$ $\frac{16}{64}$	$P(Y=1)$ $\frac{3}{32}$	$P(Y=2)$ $\frac{3}{32}$	$P(Y=3)$ $\frac{11}{64}$	$P(Y=4)$ $\frac{13}{64}$	$P(Y=5)$ $\frac{6}{32}$	1

(i) $P(X \leq 1) = P(X=0) + P(X=1)$

$$= \frac{8}{32} + \frac{10}{16}$$

$$= \frac{28}{32} = \frac{7}{8}$$

(ii) $P(Y \leq 3) = P(Y=1) + P(Y=2) + P(Y=3)$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64}$$

$$= \frac{23}{64}$$

(iii) $P(X \leq 1, Y \leq 3) = P(0,1) + P(0,2) + P(0,3) + P(1,1) + P(1,2) + P(1,3)$

$$= 0 + 0 + \frac{1}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8}$$

$$= \frac{9}{32}$$

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(iv) $P(X \le 1 \mid Y \le 3) = \dfrac{P((X \le 1) \cap P(Y \le 3))}{P(Y \le 3)}$

$= \dfrac{9/32}{\frac{23}{64}}$

$= \dfrac{18}{23}$

(v) $P(Y \le 3 / X \le 1) = \dfrac{P(Y \le 3 \cap X \le 1)}{P(X \le 1)}$

$= \dfrac{9/32}{7/8}$

$= 9/28.$

(vi) $P(X+Y \le 4) = P(0,1) + P(0,2) + P(0,3) + P(0,4)$

$+ P(1,1) + P(1,2) + P(1,3) +$

$P(2,1) + P(2,2)$

$= 0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8}$

$+ \frac{1}{32} + \frac{1}{32}$

$= \frac{13}{32}$

4. The joint probability mass function of X and Y is

Y \ X	0	1	2
0	0.10	0.04	0.02
1	0.08	0.20	0.06
2	0.06	0.14	0.30

Find the marginal distribution function of X & Y. Also $P(X \leq 1, Y \leq 1)$ and check if X and Y are independent.

Solution: MDF of X & Y

Y \ X	0	1	2	P(Y=y)
0	0.10	0.04	0.02	0.16
1	0.08	0.20	0.06	0.34
2	0.06	0.14	0.30	0.5
P(X=x)	0.24	0.38	0.32	1

$$P(X \leq 1, Y \leq 1) = P(0,0) + P(0,1) + P(1,0) + P(1,1)$$

$$= 0.10 + 0.04 + 0.08 + 0.20$$

$$= 0.42$$

To check X & Y are independent

$$P(X=0) \cdot P(Y=0) = (0.16)(0.24)$$

$$= 0.038$$

3\. **Joint Probability distribution function for continuous random variable.**

$$F[x,y] = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dx\,dy.$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$$

Marginal distribution functions:

$$F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx.$$

$$F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy.$$

Joint Probability density function:

$$f_{XY}(x,y) = \frac{\partial^2 F[x,y]}{\partial x\,\partial y}.$$

Marginal Probability density function.

$$f(x) = f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$f(y) = f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx.$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

Conditional probability density function.

$$f(y/x) = \frac{f(x,y)}{f(x)}, \quad f(x) > 0$$

$$f(x/y) = \frac{f(x,y)}{f(y)}, \quad f(y) > 0.$$

5) Show that the function ~~f(x,y)=~~

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y), & 0<x<1,\ 0<y<1 \\ 0 & \text{Otherwise.} \end{cases}$$

Find JDF of X & Y.

Solution:

(i) $f(x,y) \geq 0$ in $0 \leq x, y \leq 1$

(ii) $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy.$

$$= \int_0^1\int_0^1 \frac{2}{5}(2x+3y)\, dx\, dy.$$

$$= \frac{2}{5}\int_0^1 \cancel{\left[\frac{2x^2}{2}\right]_0^1 dy} \left[\frac{2x^2}{2} + 3xy\right]_0^1 dy$$

$$= \frac{2}{5}\int_0^1 1+3y\, dy$$

$$= \frac{2}{5}\left[y + \frac{3y^2}{2}\right]_0^1$$

$$= \frac{2}{5}\left(\left[1+\frac{3}{2}\right] - 0\right)$$

6. The Joint p.d.f of random Variable X and Y is given by

$$f(x,y) = kxy\, e^{-(x^2+y^2)}, \quad x>0,\ y>0$$

Find the Value of k. and prove also that X and Y are independent.

Solution:

$$F[x,y] = kxy\, e^{-(x^2+y^2)}$$

$$\int_0^\infty \int_0^\infty kxy\, e^{-(x^2+y^2)}\, dx\, dy = 1$$

$$k \int_0^\infty x\, e^{-(x^2)}\, dx \int_0^\infty y\, e^{-y^2}\, dy = 1$$

$$K \cdot \frac{1}{2} \cdot \frac{1}{2} = 1$$

$$\frac{K}{4} = 1$$

$$\boxed{K = 4}$$

To prove X and Y independent:

(ie) $f(x) \cdot f(y) = f(x,y)$

$$f(x) = f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_{-\infty}^{\infty} kxy\, e^{-(x^2+y^2)}\, dy$$

$$= 4xe^{-x^2}\int_0^\infty y e^{-y^2}\,dy$$

$$= \cancel{4}^{2} x e^{-x^2}\cdot\frac{1}{2}$$

$$f(x) = 2xe^{-x^2}$$

$$f(y) = f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^\infty Kxy\, e^{-(x^2+y^2)}\,dx$$

$$= 4ye^{-y^2}\int_0^\infty xe^{-x^2}\,dx$$

$$= 4ye^{-y^2}\cdot\frac{1}{2}$$

$$f(y) = 2ye^{-y^2}$$

$$f(x).f(y) = 2xe^{-x^2}.2ye^{-y^2}$$

$$= 4xye^{-(x^2+y^2)}$$

$$= f(x,y)$$

X & Y are independent

7. Let X and Y have JDF,

$f(x,y) = 2 \quad 0 < x < y < 1$. Find the MDF. Find the CDF of $(Y/X=x)$

Solution :.

M.D.f of X

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{x}^{1} 2\,dy.$$

$$= 2\,[y]_x^1$$

$$f(x) = 2\,[1-x)$$

M.d.f of Y,

$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_{0}^{y} 2\,dx$$

$$= 2\,[x]_0^y$$

$$= 2y$$

The c.d.f of Y given $X=x$ is

$$f(y/x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{2}{2(1-x)}$$

$$= \frac{1}{}$$

8. The jdf of the random Variable X and Y is given by

$$f(x,y) = \begin{cases} 8xy, & 0<x<1, \; 0<y<x \\ 0, & \text{otherwise.} \end{cases}$$

find (i) $f_X(x)$

(ii) $f_Y(y)$

(iii) $f(Y/x)$

(i) $f_X(x) = f(x) = \int_0^x 8xy \, dy$

$= 8x \int_0^x y \, dy$

$= 8x \left[\frac{y^2}{2} \right]_0^x$

$= 8x \cdot \frac{x^2}{2}$

$= 4x^3$

$f(x) = 4x^3$

(ii) $f_Y(y) = f(y) = \int_0^1 8xy \, dx$

$= 8y \left[\frac{x^2}{2} \right]_0^1$

$= 8y \cdot \frac{1}{2}$

$= 4y.$

$f(y) = 4y.$

(iii) $f(y/x) = \frac{f(x,y)}{f(x)}$

$\cancel{f(x,y)} = \frac{8\cancel{x}y^2}{\cancel{4}y} = \frac{8xy}{4x^{3}2}$

$\therefore f(y/x) = \frac{2y}{x^2}$

7. The Joint p.d.f of a two dimensional random Variable (X,Y) is given by.

$f(x,y) = xy^2 + x^2/8 \; , \; 0 \le x \le 2$
$0 \le y \le 1$

Compute (i) $P(X>1 \,/\, Y<1/2)$

(ii) $P(Y<1/2 \,/\, X>1)$

(iii) $P(X<Y)$

(iv) $P(X+Y \le 1)$

Solution :-

(i) $P(X>1/Y<1/2) = \frac{P(X>1, Y<1/2)}{P(Y<1/2)}$

$P(X>1, Y<1/2) = \int_1^2 \int_0^{1/2} \left(xy^2 + \frac{x^2}{8}\right) dy\, dx$

$= \int_1^2 \left[\frac{xy^3}{3} + \frac{x^2 y}{8}\right]_0^{1/2} dx$

$= \int_1^2 \left[\left[\frac{x}{24} + \frac{x^2}{16}\right] - (0+0)\right] dx$

$$= \int_1^2 \left(\frac{x}{24} + \frac{x^2}{16} \right) dx$$

$$= \left[\frac{x^2}{48} + \frac{x^3}{48} \right]_1^2$$

$$= \frac{1}{48} \left[(4+8) - (1+1) \right]$$

$$= \frac{1}{48} \left[12 - 1 \right]$$

$$= \frac{10}{48} = \frac{5}{24}$$

$$P(Y < 1/2) = \int_0^{1/2} \int_0^2 \left(x y^2 + \frac{x^2}{8} \right) dx\, dy$$

$$= \cancel{\left[\frac{xy^3}{3} + \frac{x^2 y}{8} \right]_0^{1/2}}$$

$$= \cancel{\frac{x}{24}} + \cancel{\frac{x^2}{16}}$$

$$= \int_0^{1/2} \left[\frac{x^2 y^2}{2} + \frac{x^3}{24} \right]_0^2 dy$$

$$= \int_0^{1/2} \left(\left[\frac{4y^2}{2} + \frac{8}{24} \right] - \left[0 + 0 \right] \right) dy$$

$$= \int_0^{1/2} \left[\frac{4y^2}{2} + \frac{1}{3} \right] dy$$

$$= \left[\frac{2}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{3}\left(\frac{1}{2}\right)\right]$$

$$= \left[\frac{2}{24} + \frac{1}{6}\right]$$

$$= \frac{6}{24} = \frac{1}{4}$$

$$P\left[X>1 / Y<\tfrac{1}{2}\right] = \frac{P(X>1,\ Y<\tfrac{1}{2})}{P(Y<\tfrac{1}{2})}$$

$$= \frac{5/24}{1/4}$$

$$= 5/6$$

$\therefore\ P\left[X>1 / Y<\tfrac{1}{2}\right] = 5/6.$

(ii) ~~$P(X>1 / Y<\tfrac{1}{2})$~~

$$P(Y<\tfrac{1}{2} / X>1) = \frac{P(Y<\tfrac{1}{2};\ X>1)}{P(X>1)}$$

$$P(X>1) = \int_0^1 \int_1^2 \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \int_0^1 \left(\frac{x^2}{2}y^2 + \frac{x^3}{24}\right)_1^2 dy.$$

$$= \int_0^1 \left[\left[\frac{4y^2}{2} + \frac{8}{24}\right] - \left[\frac{y^2}{2} + \frac{1}{24}\right)\right] dy$$

$$= \left[\frac{2y^3}{3} + \frac{y}{3} - \frac{y^3}{6} - \frac{y}{24}\right]_0^1$$

$$= \left[\frac{2}{3} + \frac{1}{3} - \frac{1}{6} - \frac{1}{24}\right]$$

$$= \left[\frac{3}{3} - \frac{1}{6} - \frac{1}{24}\right]$$

$$= \left[\frac{24-4-1}{24}\right]$$

$$= \frac{19}{24}$$

$$P(Y<1/2 / X>1) = \frac{P(Y<1/2, X>1)}{P(X>1)}$$

$$= \frac{5/24}{19/24}$$

$$= \frac{5}{19}$$

$$\therefore P[Y<1/2 / X>1] = \frac{5}{19}$$

(iii) $$P(X<Y) = \int_0^1 \int_0^y \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \int_0^1 \left(\frac{x^2y^2}{2} + \frac{x^3}{24}\right)_0^y dy$$

$$= \int_0^1 \left(\frac{y^4}{2} + \frac{y^3}{24}\right) dy$$

$$= \left[\frac{y^5}{} + \frac{y^4}{}\right]^1$$

Covariance:

If X and Y are random variables, then covariance between X and Y is defined as

$$Cov(X,Y) = E[XY] - E[X].E[Y].$$

Note:

If X and Y are independent then, $E[XY] = E[X].E[Y]$

$\Rightarrow Cov(X,Y) = 0$

(1) $Cov(aX, bY) = ab\, cov(X,Y)$

(2) $Cov(X+a, Y+b) = Cov(X,Y)$

(3) $Cov(aX+b, cY+d) = ac\, cov(X,Y)$

(4) $Var(X_1+X_2) = Var\, X_1 + Var\, X_2 + 2\, cov(X_1, X_2)$

(5) $Var(X_1-X_2) = Var\, X_1 + Var\, X_2 - 2\, cov(X_1, X_2)$

(6) If X_1 & X_2 are independent, then

$Var(X_1 \pm X_2) = Var\, X_1 \pm Var\, X_2.$

Correlation:

Karl-pearsons Coefficient of Correlation:

$$r(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

where,

$$\sigma_X = \sqrt{\frac{1}{n}\left\{\Sigma x^2 - \bar{x}^2\right\}}, \quad \bar{X} = \frac{\Sigma x}{n}$$

$$\sigma_Y = \sqrt{\frac{1}{n}\Sigma y^2 - \bar{y}^2}, \quad \bar{Y} = \frac{\Sigma y}{n}$$

1) Correlation Coefficient may also be denoted by $\rho(x,y)$ (or) ρ_{XY}

2) If $\rho(x,y) = 0$, we say that x and y are uncorrelated.

3) When $r=1$, the correlation is perfect and positive.

4) Two independent variables are uncorrelated. Since $Cov(x,y)=0$ when X and Y are independent

$$r(x,y) = \frac{Cov(x,y)}{\sigma_X \sigma_Y} = 0$$

10.

Q1. Calculate the Correlation Coefficient for the following ~~types~~ height (in inches) of fathers (x) and their sons (y)

X	65	66	67	67	68	69	70	72
Y	67	68	65	68	72	72	69	71

Solution:.

X	Y	XY	X^2	Y^2
65	67	4355	4225	4489
66	68	4488	4356	4624
67	65	4355	4489	4225
67	68	4556	4489	4624
68	72	4896	4624	5184
69	72	4968	4761	5184
70	69	4830	4900	4761
72	71	5112	5184	5041
$\Sigma x =$ 544	$\Sigma y =$ 552	$\Sigma xy =$ 37560	$\Sigma x^2 =$ 37028	$\Sigma y^2 =$ 38132

$$\bar{X} = \frac{\Sigma x}{n} = \frac{544}{8} = 68$$

$$\bar{Y} = \frac{\Sigma y}{n} = \frac{552}{8} = 69$$

$$\cancel{\overline{\Sigma x}^2 = \frac{\Sigma x^2}{n} = \frac{37028}{8} =}$$

$$\bar{x}\,\bar{y} = 68 \times 69 = 4692$$

$$\sigma_x = \sqrt{\frac{1}{n}\Sigma x^2 - \bar{x}^2}$$

$$= \sqrt{\frac{1}{8}(37028) - (68)^2}$$

$$= 2.121$$

$$\sigma_Y = \sqrt{\frac{1}{n}\Sigma Y^2 - \bar{Y}^2}$$

$$= \sqrt{\frac{1}{8}(38132) - 69^2}$$

$$= 2.345$$

$$r(X,Y) = \frac{Cor(X,Y)}{\sigma_X \sigma_Y}$$

$$= \frac{\frac{1}{n}\Sigma XY - \bar{X}\bar{Y}}{\sigma_X \sigma_Y}$$

$$= \frac{\frac{1}{8}(37560) - 4692}{2.121 \times 2.345}$$

$$= 0.6030.$$

2. Find the correlation coefficient for the following data.

X	10	14	18	22	26	30
Y	18	12	24	6	30	36

Solution:

X	Y	$U = \frac{X-22}{4}$	$V = \frac{Y-24}{6}$	UV	U^2	V^2
10	18	-3	-1	3	9	1
14	12	-2	-2	4	4	4
18	24	-1	0	0	1	0
22	6	0	-3	0	0	9
26	30	1	1	1	1	1
30	36	2	2	4	4	4

$$\bar{u} = \frac{\Sigma u}{n} = \frac{-3}{6} = -0.5$$

$$\vec{v} = \frac{\Sigma v}{n} = \frac{-3}{6} = -0.5$$

$$\bar{U}\bar{V} = -0.5 \times -0.5$$

$$= 0.25$$

$$\sigma_u = \sqrt{\frac{1}{6}\Sigma u^2 - (\bar{u})^2}$$

$$= \sqrt{\frac{1}{6}(19) - (0.25)}$$

$$= 1.70$$

$$\sigma_v = \sqrt{\frac{1}{6}\Sigma v^2 - (\bar{v})^2}$$

$$= \sqrt{\frac{1}{6}(19) - (0.25)}$$

$$= 1.70$$

$$r(x,y) =$$

$$r(u,v) = \frac{Cov(u,v)}{\sigma_u \sigma_v}$$

$$= \frac{\frac{1}{n}\Sigma uv - \bar{u}\bar{v}}{\sigma_u \sigma_v}$$

$$= \frac{\frac{1}{6}(12) - 0.25}{(1.70)(1.70)}$$

$$= \frac{1.75}{2.917266} = 0.6$$

3. The Joint probability mass Function x and Y is

Y / X	-1	1
0	1/8	3/8
1	2/8	2/8

Find the correlation coefficient of x and Y.

Solution:

Y / X	-1	1	P(X=x)
0	1/8 P(0,-1)	3/8 P(0,1)	4/8 P(x=0)
1	2/8 P(1,-1)	2/8 P(1,1)	4/8 P(X=1)
P(Y=y)	3/8 P(Y=-1)	5/8 P(Y=1)	1

$E[x] = \sum x\, P(x)$

$= 0 \times \frac{1}{2} + 1 \times \frac{1}{2}$

$= \frac{1}{2}$

$E[Y] = \sum y\, P(y)$

$= -1 \times \frac{3}{8} + 1 \times \frac{5}{8}$

$= \frac{1}{4}$

$E[x^2] = \sum x^2\, P(x)$

$$E[Y^2] = \sum y^2 P(y)$$

$$= (-1)^2 \times \frac{3}{8} + 1^2 \times \frac{5}{8}$$

$$= 1.$$

$$E[XY] = \sum xy \, p(x,y)$$

$$= 0 \times -1 \times \frac{1}{8} + 0 \times 1 \times \frac{3}{8} + 1 \times -1 \times \frac{2}{8} + 1 \times 1 \times \frac{2}{8}$$

$$= 0$$

$$\sigma_X^2 = E[X^2] - [E[X]]^2$$

$$= \frac{1}{2} - \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{4}$$

$$\sigma_Y^2 = E[Y^2] - [E(Y)]^2$$

$$= 1 - \left(\frac{1}{4}\right)^2$$

$$= \frac{15}{16}$$

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y}$$

$$= \frac{0 - \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)}{\sqrt{\frac{1}{4}}\sqrt{\frac{15}{16}}}$$

17) If the independent random Variable X and Y have the Variance 36 and 16 respectively. Find the correlation Coefficients b/w X+Y and X−Y.

Solution!.

Given :

$Var(X) = 36$

$Var(Y) = 16$

X+Y are independent.

$E[XY] = E[X]\,E[Y]$

Let $u = X+Y$

$V = X-Y$

$Var(u) = Var(X+Y)$

$= 1^2\,Var(X) + 1^2\,Var(Y)$

$= 1\times 36 + 1\times 16 = 36+16$

$= 52.$

$\sigma_u^2 = 52$

$Var(V) = Var(X-Y)$

$= 1^2\,Var(X) + (-1)^2\,Var(Y)$

$= 1\times 36 + 1\times 16$

$= 36+16$

$= 52$

$\sigma_V^2 = 52$

$\sigma_u = \sqrt{52}$, $\sigma_V = \sqrt{52}$

$$Cov(u,v) = E[uv] - E[u]E[v]$$

$$E[uv] = E[(x+y)(x-y)]$$
$$= E[x^2 - y^2]$$

$$E[uv] = E[x^2] - E[y^2]$$

$$E[u] = E[x+y]$$
$$= E[x] + E[y]$$

$$E[v] = E[x-y]$$
$$= E[x] - E[y]$$

$$Cov(u,v) = E[x^2] - E[y^2] - [[E[x]]^2 - [[E(y)]]^2]$$
$$= E[x^2] - E[y^2] - [(E[x])^2] + [(E(y))^2]$$
$$= [E[x^2] - [(E[x])^2]] -$$
$$[E[y^2] - (E(y))^2]$$

$$= Var(x) - Var(y)$$
$$= 36 - 16$$
$$= 20.$$

$$\rho(u,v) = \frac{Cov(u,v)}{\sigma_u \cdot \sigma_v}$$
$$= \frac{20}{\sqrt{52} \cdot \sqrt{52}}$$
$$= \frac{20}{52}$$

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If the joint p.d.f of (X,Y) is given by $f(x,y) = x+y$, $0 \le x, y \le 1$

Find P_{xy}.

Solution:

Given

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \ f(x,y)\, dx\, dy$$

$$= \int_0^1\int_0^1 xy\,(x+y)\, dx\, dy$$

$$= \int_0^1\int_0^1 x^2y + xy^2 \ dx\, dy$$

$$= \int_0^1 \left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_0^1 dy$$

$$= \int_0^1 \left(\frac{y}{3} + \frac{y^2}{2}\right) dy$$

$$= \left[\frac{y^2}{6} + \frac{y^3}{6}\right]_0^1$$

$$= \frac{1}{6} + \frac{1}{6}$$

$$= 2/6$$

$$= 1/3$$

$E[X,Y] = 1/3$.

Marf of X, $f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$

$= \int^1 \ldots$

$$= \left[xy + \frac{y^2}{2}\right]_0^1$$

$$= xy + \frac{1}{2}$$

$f(x) = x + 1/2$

Malf of Y, $f(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$

$$= \int_0^1 (x+y)\, dx$$

$$= \left[\frac{x^2}{2} + xy\right]_0^1$$

$$= \frac{1}{2} + y$$

$f(y) = y + 1/2$

$E[x] = \int_{-\infty}^{\infty} x\, f(x)\, dx.$

$$= \int_0^1 x\,(x + 1/2)\, dx$$

$$= \int_0^1 x^2 + \frac{x}{2}\, dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{4}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{12}.$$

$$E[Y] = \int_{-\infty}^{\infty} y \, f(y) \, dy$$

$$= \int_{0}^{1} y \left(y + \tfrac{1}{2}\right) dy$$

$$= \int_{0}^{1} \left(y^2 + \frac{y}{2}\right) dy$$

$$= \left[\frac{y^3}{3} + \frac{y^2}{4}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{12}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x) \, dx$$

$$= \int_{0}^{1} x^2 \left(x + \tfrac{1}{2}\right) dx$$

$$= \int_{0}^{1} \left(x^3 + \frac{x^2}{2}\right) dx$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{6}\right]_0^1$$

$$= \frac{1}{4} + \frac{1}{6}$$

$$= \frac{10}{24}$$

$$= 5/12$$

$$E[Y^2] = \int_{-\infty} y^2 \, f(x) \, d\ldots$$

$$= \int_0^1 \left(y^3 + \frac{y^2}{2}\right) dy$$

$$= \left[\frac{y^4}{4} + \frac{y^3}{6}\right]_0^1$$

$$= \frac{1}{4} + \frac{1}{6}$$

$$= \frac{10}{24}$$

$$= 5/12.$$

$$Var(X) = E[X^2] - [E(X)]^2$$

$$= \frac{5}{12} - \frac{49}{144}$$

$$\sigma_x^2 = \frac{11}{144} \Rightarrow \sigma_x = \sqrt{\frac{11}{144}}$$

$$Var(Y) = E[Y] - (E(Y))^2$$

$$= \frac{7}{12} - \cancel{\frac{25}{144}}\ \frac{49}{144}$$

$$\sigma_y^2 = \frac{\cancel{5}9}{144} \Rightarrow \sigma_Y = \cancel{\sqrt{\frac{59}{144}}}\ \sqrt{\frac{11}{144}}$$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$= \frac{E[XY] - E[X]\,E[Y]}{\sigma_X \sigma_Y}$$

$$= \frac{1}{3} - \frac{7}{12} \cdot \frac{7}{12}$$

$$= \frac{\frac{1}{3} - \frac{49}{144}}{\frac{11}{144}}$$

$$= \frac{48-49}{144} \times \frac{144}{11}$$

$$= \frac{-1}{11}$$

$$\therefore P(x,y) = \frac{-1}{11}$$

If $f(x,y) = \frac{6-x-y}{8}$, $0 \le x \le 2$, $2 \le y \le 4$, find the correlation coefficient between X and Y.

Solution:

Given

$$f(x,y) = \frac{6-x-y}{8} \quad \begin{matrix} 0 \le x \le 2 \\ 2 \le y \le 4 \end{matrix}$$

Mdf of x:

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_{2}^{4} \frac{6-x-y}{8}\, dy = \frac{1}{8}\left[6y - xy - y^2/2\right]_{2}^{4}$$

$$= \left[\frac{6y - xy - \cancel{y^2}}{16}\right]_{2}^{4} = \frac{1}{8}\left[\cancel{24-4x-16}\right]$$

$$= \left[\frac{24 - 4x - 16}{\cancel{16}}\right] - \left[\frac{12 - 2x - 4}{\cancel{16}}\right]$$

$$= \frac{1}{8}\left[\left(24 - 4x - \frac{16}{x}\right) - \left(12 - 2y - 4/x\right)\right]$$

$$= \frac{1}{8}\left[16 - 4x + 10 + 2y\right]$$

$$= \frac{1}{8}\left[6 - 2x\right]$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_0^2 \frac{6 - x - y}{8}\, dx$$

$$= \frac{1}{8}\int_0^2 6 - x - y \, dx$$

$$= \frac{1}{8}\left[6x - \frac{x^2}{2} - xy\right]_0^2$$

$$= \frac{1}{8}\left[\left[72 - \frac{144}{2} - 12y\right] - [0]\right]$$

$$= \frac{1}{8}\left[72 - 72 - 12y\right]$$

$$= \cancel{-\frac{12y}{8}}$$

$$= \cancel{-3y/2}$$

$$= \frac{1}{8}\left[12 - \frac{4}{2} - 2y\right]$$

$$= \frac{1}{8}\left[10 - 2y\right]$$

$$= \frac{1}{4}(5 - y).$$

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$$E[X \cdot Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f(x,y)\, dx\, dy$$

5/6

$$= \int_{2}^{4}\int_{0}^{2} xy\left(\frac{6-x-y}{8}\right) dx\, dy$$

$$= \frac{1}{8}\int_{2}^{4}\int_{0}^{2} 6xy - x^2y - xy^2 \, dx\, dy$$

$$= \frac{1}{8}\int_{2}^{4}\left[\frac{6x^2y}{2} - \frac{x^3y}{3} - \frac{x^2y^2}{2}\right]_0^2 dy$$

$$= \frac{1}{8}\int_{2}^{4}\left[\frac{6(4)y}{2} - \frac{8y}{3} - \frac{4y^2}{2}\right] dy$$

$$= \frac{1}{8}\int_{2}^{4} 12y - \frac{8y}{3} - 2y^2 \, dy$$

$$= \frac{1}{8}\left[\frac{12y^2}{2} - \frac{8y^2}{6} - \frac{2y^3}{3}\right]_2^4$$

$$= \frac{1}{8}\left\{\left[\frac{12\times 16}{2} - \frac{8(16)}{6} - \frac{2(64)}{3}\right] - \left[\frac{12\times 4}{2} - \frac{8\times 4}{6} - \frac{2}{}\right.\right.$$

$$= \frac{1}{8}\left\{\left[96 - \frac{64}{3} - \frac{128}{3}\right] - \left[24 - \frac{16}{3} - \frac{16}{3}\right]\right\}$$

$$= \frac{1}{8}\left[32 - \frac{40}{3}\right]$$

$$= 4 - \frac{5}{3} = \frac{12-5}{3} = \frac{7}{3}$$

$$E[X] = \int_{-\infty}^{\infty} x \, f(x) \, dx$$

$$\cancel{= \int_0^2 x \left(\frac{6-x-y}{8} \right) dx}$$

$$\cancel{= \frac{1}{8} \int_0^2 6x - x^2 - xy \, dx}$$

$$\cancel{= \frac{1}{8} \left[\frac{6x^2}{2} - \frac{x^3}{3} - \frac{x^2 y}{2} \right]_0^2}$$

$$\cancel{= \frac{1}{8} \left[\frac{6 \times 4}{2} - \frac{8}{3} - \frac{4y}{2} \right]}$$

$$\cancel{= \frac{1}{8} \left[12 - \frac{8}{3} - y \right]}$$

$$= \int_0^2 x \left(\frac{1}{8} (6 - 2x) \right) dx$$

$$= \frac{1}{8} \int_0^2 6x - 2x^2 \, dx$$

$$= \frac{1}{8} \left[\frac{6x^2}{2} - \frac{2x^3}{3} \right]_0^2$$

$$= \frac{1}{8} \left[\frac{6(4)}{2} - \frac{2(8)}{3} \right]$$

$$= \frac{1}{8} \left[12 - \frac{16}{3} \right]$$

$$= \frac{1}{8} \left[\frac{36 - 16}{3} \right]$$

$$= \frac{1}{8} \left[\frac{20}{3} \right]$$

$$= 5/6$$

$$E[X] = 5/6$$

$$E[Y] = \int_{-\infty}^{\infty} y \, f(y) \, dy$$

$$= \int_2^4 y \, \frac{1}{4} (5 - y) \, dy$$

$$= \frac{1}{4} \int_2^4 5y - y^2 \, dy$$

$$= \frac{1}{4}\left[\left[\frac{5\times \cancel{16}^{8}}{\cancel{2}} - \frac{64}{3}\right] - \left[\frac{5\times \cancel{4}^{2}}{\cancel{2}} - \frac{8}{3}\right]\right]$$

$$= \frac{1}{4}\left[\left[40 - \frac{64}{3}\right] - \left[10 - \frac{8}{3}\right]\right]$$

$$= \frac{1}{4}\left[\left(\frac{120-64}{3}\right) - \left(\frac{30-8}{3}\right)\right]$$

$$= \frac{1}{4}\left[\frac{56}{3} - \frac{22}{3}\right]$$

$$= \frac{1}{\cancel{4}_{2}} \times \frac{\cancel{34}^{17}}{3}$$

$$= \frac{17}{6}$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 \, f(x)\, dx$$

$$= \int_0^2 x^2 \left(\frac{6-2x}{8}\right) dx$$

$$= \frac{1}{8}\int_0^2 6x^2 - 2x^3 \, dx$$

$$= \frac{1}{8}\left[\frac{6x^3}{3} - \frac{2x^4}{4}\right]_0^2$$

$$= \frac{1}{8}\left[\frac{\cancel{6}^{2}\times 8}{\cancel{3}} - \frac{2\times \cancel{16}^{4}}{\cancel{4}}\right]$$

$$= \frac{1}{8}\left[16 - 8\right]$$

$$= 1$$

$$E[Y^2] = \int_{-\infty} y^2 \, f(y) \, dy.$$

$$= \int_2^4 y^2 \left(\frac{5-y}{4}\right) dy$$

$$= \frac{1}{4}\int_2^4 5y^2 - y^3 \, dy.$$

$$= \frac{1}{4}\left[\frac{5y^3}{3} - \frac{y^4}{4}\right]_2^4$$

$$= \frac{1}{4}\left[\left[\frac{5\times 64}{3} - \frac{\cancel{256}^{64}}{\cancel{4}}\right] - \left[\frac{5\times 8}{3} - \frac{\cancel{16}^{4}}{\cancel{4}}\right]\right]$$

$$= \frac{1}{4}\left[\left[\frac{320}{3} - 64\right] - \left[\frac{40}{3} - 4\right]\right]$$

$$= \frac{1}{4}\left[\frac{280}{3} - 60\right]$$

$$= \frac{1}{4}\left[\frac{280-180}{3}\right]$$

$$= \frac{1}{\cancel{4}}\left(\frac{\cancel{100}^{25}}{3}\right)$$

$$= 25/3$$

$$Var(x) = \sigma_x^2 = E[x^2] - (E(x))^2$$

$$= 1 - (5/6)^2$$

$$= 1 - \frac{25}{36}$$

$$= \frac{36-25}{}$$

$$\sigma_x = \sqrt{\frac{11}{36}}$$

$$Var(Y) = \sigma_Y^2 = E[Y^2] - (E(Y))^2$$

$$= \frac{25}{3} - \left(\frac{17}{6}\right)^2$$

$$= \frac{25}{3} - \frac{289}{36}$$

$$= \frac{300-289}{36}$$

$$= \frac{11}{36}$$

$$\sigma_Y = \sqrt{11/36}$$

$$P(X,Y) = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}$$

$$= \frac{E[XY] - E[X]E[Y]}{\sigma_X \cdot \sigma_Y}$$

$$= \frac{\frac{7}{3} - \frac{5}{6} \cdot \frac{17}{6}}{\sqrt{\frac{11}{36}} \cdot \sqrt{\frac{11}{36}}}$$

$$= \frac{\frac{7}{3} - \frac{85}{36}}{\frac{11}{36}}$$

$$= \frac{84-85}{\cancel{36}} \times \frac{\cancel{36}}{11}$$

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Correlation Coefficient:

$$r = \pm\sqrt{b_{XY} \times b_{YX}}$$

$$b_{YX} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2}$$

$$b_{XY} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2}$$

1. From the following data, Find

(i) The two regression eqn.

(ii) The Coefficient of correlation between the marks in economics and Statistics.

(iii) The most likely marks in Statistics, when marks in economic 30.

Marks in economics	25	28	35	32	31	36	29	38	34	3
Marks in Statistics	43	46	49	41	36	32	31	30	33	3

$\cancel{b_{yx} = 0.6643}$ $\boxed{b_{yx} = -0.6643}$

$$b_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2}$$

$$= -\frac{93}{398}$$

$\cancel{① \Rightarrow y =}$ -0.2337

$$\boxed{b_{xy} = -0.2337}$$

① $\Rightarrow y - 38 = -0.6643(x-32)$

$y - 38 = -0.6643x \times (-0.6643x - 32)$

$y = -0.6643x \times 21.2576 + 38$

$y = -0.6643x \; 59.2576$

② $\Rightarrow x - 32 = -0.2337(y-38)$

$x = -0.2337y \times (-0.2337 \times -38) + 32$

$= -0.2337y \times 8.8806 + 32$

$x = 40.8806 - 0.2337y.$

(ii) Coefficient of Correlation

$$r = \pm\sqrt{b_{yx} \times b_{xy}}$$

$$= \pm\sqrt{(-0.6643)(-0.2337)}$$

$$= \pm\sqrt{0.152}$$

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The tangent of the angle between the lines of regression of Y on X and X on Y is 0.6 and $\sigma_x = \frac{1}{2}\sigma_y$. Find the correlation coefficient between X and Y.

$\tan\theta = 0.6$ $\sigma_x = 0.5\sigma_y$

Solution:

Given:

$$\tan\theta = 0.6.$$

$$\sigma_x = 0.5\,\sigma_y.$$

Angle b/w lines of regression is

$$\tan\theta = \frac{1-r^2}{r}\left(\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}\right)$$

$$0.6 = \frac{1-r^2}{r}\left(\frac{(0.5\sigma_y)\,\sigma_y}{(0.5\sigma_y)^2+\sigma_y^2}\right)$$

$$= \frac{1-r^2}{r}\left(\frac{0.5\sigma_y^2}{0.25\sigma_y^2+\sigma_y^2}\right)$$

$$0.6 = \frac{1-r^2}{r}\left(\frac{0.5\cancel{\sigma_y^2}}{1.25\cancel{\sigma_y^2}}\right)$$

$$\frac{1-r^2}{r} = \frac{(1.25)(0.6)}{0.5}$$

$$= \frac{0.75}{0.5}$$

$r^2 + 1.5r - 1 = 0$

$r = \frac{1}{2}, -2$ (−2 is not possible)

$\boxed{r = 1/2}$

19/2/2013 Transformation of two dimensional random Variable:.

$$f_{uv} = f_{xy}(x,y) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

$$f_u(u) = \int_{-\infty}^{\infty} f_{uv}(u,v)\, dv$$

$$f_v(u,v) = \int_{-\infty}^{\infty} f_{uv}(u,v)\, du.$$

If the joint p.d.f of X, Y is given by $f_{XY}(x,y) = x+y$, $0 \le x, y \le 1$. Find the p.d.f of $U = XY$.

Solution:

Step 1:

To find joint p.d.f of X & Y.

Given:

$f_{XY}(x,y) = x+y.$

Step 2:

Introdu...

Step 3 :

Expressing the above eqn as

$x = g_1(u,v)$ $\quad$ $y = g_2(u,v)$

$u = xy$ $\quad$ $v = y$

$u = x \cdot v$ $\quad$ $y = v$

$x = \frac{u}{v}$

$\frac{\partial x}{\partial u} = \frac{1}{v}$ $\quad$ $\frac{\partial x}{\partial v} = \frac{-u}{v^2}$

$\frac{\partial y}{\partial u} = 0$ $\quad$ $\frac{\partial y}{\partial v} = 1$

Step 4 :

Find $|J| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$

$$J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ 0 & 1 \end{vmatrix}$$

$$= \frac{1}{v} - 0$$

$$= \frac{1}{v}$$

$$|J| = \frac{1}{v}$$

Step 5 :

To find pdf of (u,v)

$f_{uv}(u,v) = f_{xy}(x,y)\, |J|$

$$= \left(\frac{u}{v} + v\right) \cdot \frac{1}{v}$$

Step 6:- Changing the domain, (x, y) into domain (u, v)

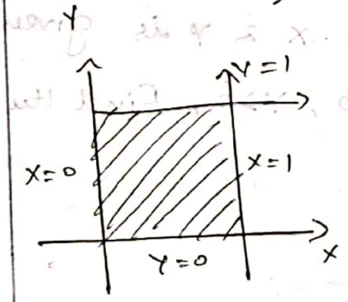
$0 \le y \le 1 \Rightarrow 0 \le v \le 1$

$0 \le x \le 1 \Rightarrow 0 \le \frac{u}{v} \le 1$

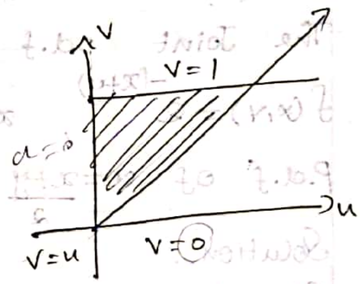
$\Rightarrow 0 \le u \le v$

$u = v$

u	0	1	2	3
v	0	1	2	3



(x, y)



(u, v)

P.d.f of (u, v) is given by

$$f_{uv}(u, v) = \frac{1}{v}\left(\frac{u}{v} + v\right) \quad 0 \le u \le v, \; 0 \le v \le 1$$

Step 7:

To find the p.d.f. of $(u = xy)$

$$f_u(u) = \int_{-\infty}^{\infty} f_{uv}(u, v)\, dv$$

$$= \int_u^1 \frac{1}{v}\left(\frac{u}{v} + v\right) dv$$

$$= \int_u^1 \frac{u}{v^2} + 1 \, dv$$

$$= \left[u . \frac{v^{-1}}{} + v \right]^1$$

$= \left[(-u+1) - (-1+u)\right]$ $\quad -u+1+1-u$

$\quad 2-2u$

$= -u+1+1-u$ $\quad 2(1-u)$

$= 2-2u$

$= 2(1-u)$

$f_U(u) = 2(1-u) \quad 0 \le u \le 1$

The Joint pd.f of X & Y is given by $f(x,y) = e^{-(x+y)}$, $x>0$, $y>0$, Find the P.d.f of $u = \frac{x+y}{2}$

Solution:

Step 1:

To find Joint pdf of X, Y.

Given

$f(x,y) = e^{-(x+y)}$, $x>0$, $y>0$.

Step 2:

Introducing new random Variable.

Given, $u = \frac{x+y}{2}$

Let $v=y$.

Step 3:

Expressing the above eqn as

$x = g_1(u,v)$ & $y = g_2(u,v)$

$u = \frac{x+y}{2}$ $\quad v=y$

$u = \frac{x+v}{2}$ $\quad y=v$

$2u = x+v$

$\frac{\partial x}{\partial u} = 2$ $\qquad \frac{\partial x}{\partial v} = -1$

$\frac{\partial y}{\partial u} = 0$ $\qquad \frac{\partial y}{\partial v} = 1$

Step 4:.

Find $|J| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$

$$|J| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= 2$$

Step 5:.

To find pdf of (u,v)

$$f_{UV}(u,v) = f_{XY}(x,y)\,|J|$$

$$= \cancel{(x+y)}\, e^{-(x+y)} \cdot \cancel{2}$$

$$= e^{-\{x+y\}} \cdot 2$$

$$= 2e^{-2u}$$

$x > 0$ $\qquad y > 0$

$2u - v > 0$ $\qquad v > 0$

$2u > v$

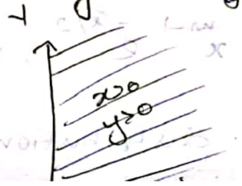
$u > v/2$

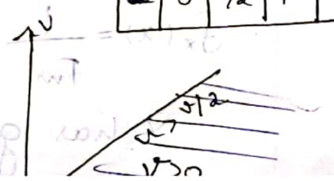
Step 6: Changing the domain (x,y) into (u,v)

$x > 0 \Rightarrow u > v/2$

$y > 0 \Rightarrow v > 0$.

v	0	1	2	3
u	0	1/2	1	1.5





Step II:-

To find the pdf of $u = \frac{x+y}{2}$

$$f_u(u) = \int_{-\infty}^{\infty} f_{uv}(u,v)\, dv$$

$$= \int_0^{2u} 2e^{-2u}\, dv$$

$$= 2e^{-2u} \int_0^{2u} dv$$

$$= 2e^{-2u} [v]_0^{2u}$$

$$= 2e^{-2u} \cdot [2u-0]$$

$$= 4ue^{-2u} \quad u>0.$$

The random Variable X and Y are Statistically independent having gamma Variable with parameters $(m, 1/2)$ and $(n, 1/2)$ respectively. Derive the p.d.f of a random variable $U = \frac{X}{X+Y}$

Solution:-

Step 1: To find Joint p.d.f of X, Y.

X has a gamma distribution with parameters $(m, 1/2)$

$$f_X(x) = \frac{1}{\Gamma m} \frac{1}{2^m} x^{m-1} e^{-x/2}, \quad x>0.$$

Y has gamma distribution with

$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$.

$$= \left[\frac{1}{\Gamma m} \cdot \frac{1}{2^m} x^{m-1} e^{-x/2}\right]\left[\frac{1}{\Gamma n} \frac{1}{2^n} y^{n-1} e^{-y/2}\right]$$

Step 2: Given Introducing new random variable.

$$u = \frac{X}{X+Y}$$

Let $V = X+Y$.

Step 3: Expressing the above eqn as

$x = g_1(u,v)$ & $y = g_2(u,v)$

$$u = \frac{x}{x+y}$$

$$u = \frac{x}{v}$$

$$x = uv$$

$$v = x+y$$

$$v = uv + y$$

$$y = v - uv$$

$$y = v(1-u)$$

$$\frac{\partial x}{\partial u} = v \qquad \frac{\partial x}{\partial v} = u$$

$$\frac{\partial y}{\partial u} = -v \qquad \frac{\partial y}{\partial v} = 1-u$$

Step 4:

To find $|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix}$$

$$= v(1-u) - (-uv)$$

$$|J| = v$$

Step 5: To find pdf of (u,v)

$$f_{UV}(u,v) = f_{XY}(x,y)\,|J|$$

$$= v\left[\frac{1}{\Gamma m}\cdot\frac{1}{2^m}\cdot x^{m-1}\cdot e^{-x/2}\right]\left[\frac{1}{\Gamma n}\frac{1}{2^n}y^{n-1}e^{-y/2}\right]$$

$$= v\left[\frac{1}{\Gamma m\Gamma n}\cdot\frac{1}{2^{m+n}}\,x^{m-1}\cdot y^{n-1}\cdot e^{-\frac{(x+y)}{2}}\right]$$

$$= v\left[\frac{1}{\Gamma m\Gamma n}\cdot\frac{1}{2^{m+n}}(uv)^{m-1}(v(1-u))^{n-1}\cdot e^{-v/2}\right]$$

$$= v\left[\frac{1}{\Gamma m\Gamma n\,2^{m+n}}\,u^{m-1}\,v^{m-1}\,v^{n-1}\,(1-u)^{n-1}\,e^{-v/2}\right]$$

$$= v\left[\frac{1}{\Gamma m\Gamma n\,2^{m+n}}\,u^{m-1}\cdot v^{m+n-2}\cdot(1-u)^{n-1}\cdot e^{-v/2}\right]$$

$$f_{UV}(u,v) = \left[\frac{1}{\Gamma m\Gamma n\,2^{m+n}}\,u^{m-1}\,v^{m+n-1}\,(1-u)^{n-1}\,e^{-v/2}\right]$$

Step 6:

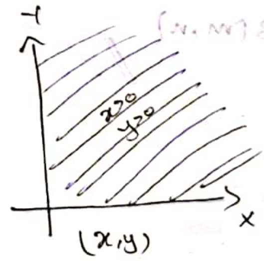
Changing the domain x,y into (u,v)

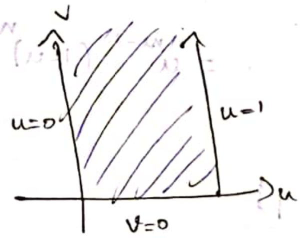
$x>0 \Rightarrow uv>0 \Rightarrow v>0.$

$y>0 \Rightarrow v-uv>0 \Rightarrow v>uv$

$\therefore \Rightarrow 1>u$

$\Rightarrow 0\le u<1$





Step I:

To find the p.d.f of $u = \frac{x}{x+y}$

$$f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)\, dv$$

$$= \int_0^\infty \frac{1}{\Gamma m \Gamma n\, 2^{m+n}} u^{m-1} v^{m+n-1} e^{-v/2} (1-u)^{n-1} \, dv$$

$$= \frac{1}{\Gamma m \Gamma n\, 2^{m+n}} u^{m-1} (1-u)^{n-1} \int_0^\infty v^{m+n-1} \cdot e^{-v/2} \, dv$$

Put $v/2 = w \Rightarrow v = 2w$

$dv = 2\,dw$

$v \to 0 \Rightarrow w \to 0$

$v \to \infty \Rightarrow w \to \infty$

$$= \frac{1}{\Gamma m \Gamma n\, 2^{m+n}} u^{m-1} (1-u)^{n-1} \int_0^\infty e^{-w} \cdot (2w)^{m+n-1}\, 2\,dw$$

$$= \frac{2 \cdot 1}{\Gamma m \Gamma n\, \cancel{2^{m+n}}} u^{m-1} (1-u)^{n-1} \int_0^\infty e^{-w} \cdot \cancel{2^{m+n}} \cdot w^{m+n-1} \, dw.$$

$$= \frac{1}{\Gamma m \Gamma n} u^{m-1} (1-u)^{n-1} \int_0^\infty e^{-w} w^{(m+n)-1} \, dw.$$

UNIT-III

Classification of Random Processes.

Stationary process:

Random Variable:

A random variable is a rule that assigns a real number to every outcome of a random experiment.

Random process.

A random process is a rule that assigns a time function to every outcome of a random experiment.

A random process is a collection of random variable $\{X(s,t)\}$ $s \in S$ (Sample space) $t \in T$ (Parameter set.

Classification of Random processes.

Discrete Random Sequence.

If both S and T are discrete then the random process is called discrete random sequence.

Eg. No. of books in library at opening time.

Continuous Random Sequence.

If 'S' is continuous and T is discrete, then the random process is

Eg: Quantity of petrol in the bulk at opening time

Discrete Random process:

If 'S' is discrete and 'T' is Continuous and the random process is called discrete random process.

Eg: No. of phone calls receiving in $(0,t)$

Continuous Random Variable Process

If 'S' is continuous and T is Continuous, then the random process is called Continuous random process

Eg. Stirring sugar in coffee

Strick Sense Stationary:

A random process is called a Strongly Stationary process (or) Strict Sense Stationary process (SSS) if all its finite dimensional distributions are invariant under translation of time parameter.

Note:

$x(t)$ is SSS

if (i) $E[x(t)]$ is Constant

(ii) $E[x^2(t)]$ is Constant.

Wide Sense Stationary:

A random process is called wide sense stationary (WSS) (or) weakly stationary process (or) Covariance stationary process.

if (i) $E[X(t)]$ is constant

(ii) Auto correlation is a function of τ (free from 't') (WSS).

Note:

A random process, non-stationary is called an evolutionary process.

$X(t)$ and $Y(t)$ are said to be Jointly WSS

(i) $R_{XY}(\tau)$ is a function of τ.

if (ii) each process is individually WSS.

22.2.2013.

1. Show that it is not-stationary. The process $X(t)$ whose probability distribution under certain conditions is given by

$$P[X(t)=n] = \begin{cases} \dfrac{(at)^{n-1}}{(1+at)^{n+1}} & n=1,2,\ldots \\ \dfrac{at}{1+at} & n=0 \end{cases}$$

Solution:

$X(t)=n$	0	1	2	...
[illegible]	at	1	(at)	

(ii) $E[x^2(t)] = \sum_{n=0}^{\infty} n^2 p(n)$

$$= 0 + \sum_{n=1}^{\infty} n^2 \left(\frac{(at)^{n-1}}{(1+at)^{n+1}}\right)$$

$$= \sum_{n=1}^{\infty} \frac{n^2 (at)^{n-1}}{(1+at)^{n-1+2}}$$

$$= \frac{1}{(1+at)^2} \sum_{n=1}^{\infty} \{n(n+1) - n\} \left(\frac{at}{1+at}\right)^{n-1}$$

$$= \frac{1}{(1+at)^2} \sum_{n=1}^{\infty} n(n+1)\left(\frac{at}{1+at}\right)^{n-1} - \sum_{n=1}^{\infty} n\left(\frac{at}{1+at}\right)^{n-1}$$

$$= \frac{1}{(1+at)^2}\left[\left[1.2\left(\frac{at}{1+at}\right)^0 + 2.3\left(\frac{at}{1+at}\right)^1 + \dots\right] - \left[1\left(\frac{at}{1+at}\right)^0 + 2\left(\frac{at}{1+at}\right) + 3\left(\frac{at}{1+at}\right)^2 + \dots\right]\right]$$

$$= \frac{1}{(1+at)^2}\left[\left[2\left(1 + 3\left(\frac{at}{1+at}\right) + 6\left(\frac{at}{1+at}\right)^2 + \dots\right] - \left[1 + 2\left(\frac{at}{1+at}\right) + 3\left(\frac{at}{1+at}\right)^2 + \dots\right]\right]$$

$$= \frac{1}{(1+at)^2}\left[2\left(1 - \frac{at}{1+at}\right)^{-3} - \left(1 - \frac{at}{1+at}\right)^{-2}\right]$$

$$= \frac{1}{(1+at)^2}\left[2\left(\frac{1+at-at}{1+at}\right)^{-3} - \left(\frac{1+at-at}{1+at}\right)^{-2}\right]$$

Given,

$X(t) = \cos(\lambda t + Y)$

$\varphi(\omega) = E[\cos \omega Y + i \sin \omega Y]$

$\varphi(1) = 0$

$\Rightarrow \varphi(1) = E[\cos Y + i \sin Y] = 0$

$E[\cos Y + i \sin Y] = 0.$

$E[\cos Y] + i\, E[\sin Y] = 0$

$E[\cos Y] = 0 \;\; \& \;\; E[\sin Y] = 0$

$\varphi(2) = 0$

$\Rightarrow E[\cos 2Y + i \sin 2Y] = 0$

$E[\cos 2Y] + i E[\sin 2Y] = 0$

$E[\cos 2Y] = 0 \;\; \& \;\; E[\sin 2Y] = 0.$

(i) $E[X(t)] = E[\cos(\lambda t + Y)]$

$= E[\cos(\lambda t) \cos y - \sin \lambda t \sin y]$

$= \cos \lambda t \; E[\cos Y] + \sin \lambda t \; E[\sin Y]$

$= 0$, constant. $\because E[\cos Y] = 0$

$E[\sin Y] = 0$

(ii) $R_{XX}(\tau) = E[X(t) \cdot X(t+\tau)]$

$= E[\cos(\lambda t + y) \cdot \cos(\lambda(t+\tau) + \cdot$

$$= E\Big[\cos\lambda t \; \cos\lambda(t+\tau) \; \cos^2 y + \sin\lambda t \; \sin\lambda(t+\tau) \; \sin^2 y + - \big[\cos\lambda t \; \sin\lambda(t+\tau) \; \cos y \cdot \sin y + \sin\lambda t \; \cos\lambda(t+\tau) \; \cos y \; \sin y\big]\Big]$$

$$= \cos\lambda t \; \cos\lambda(t+\lambda) \; E[\cos^2 y] + \sin\lambda t \; \sin\lambda(t+\tau) \; E[\sin^2 y] + [\cos\lambda t \; \sin\lambda(t+\tau) + \sin\lambda t \cdot \cos\lambda(t+\tau)] \; E[\cos y \; \sin y]$$

$$= \cos\lambda t \cdot \cos\lambda(t+\tau) \cdot E\left[\frac{1+\cos 2y}{2}\right] + \sin\lambda t \cdot \sin\lambda(t+\tau) \; E\left[\frac{1-\cos 2y}{2}\right] + E\left[\frac{\sin 2y}{2}\right]\left[\cos\lambda t \; \sin\lambda(t+\tau) + \sin\lambda t \; \cos\lambda(t+\tau)\right]$$

$$= \frac{1}{2}\left[\cos\lambda t \; \cos\lambda(t+\tau)\right] + \frac{1}{2}\left[\sin\lambda t \; \sin\lambda(t+\tau)\right]$$

$$= \frac{1}{2}\left[\cos(\lambda t - \lambda(t+\tau))\right]$$

$$= \frac{1}{2}\left[\cos(\lambda t - \lambda t - \lambda\tau)\right)$$

$$= \frac{1}{2}\cos\lambda\tau \text{ , free from } t.$$

$X(t)$ is WSS.

4. Show that the process $X(t) = A\cos\lambda t + B\sin\lambda t$.

(A, B) are random variables. is WSS.

(i) $E[A] = E[B] = 0$

(ii) $E[A^2] = E[B^2] = \varphi$

(iii) $E[AB] = 0$.

(i) $E[x(t)] = E[A\cos\lambda t + B\sin\lambda t]$

$= \cos\lambda t\, E[A] + \sin\lambda t\, E[B]$.

$= 0$, constant.

(ii) $R_{xx}(\tau) = E[x(t) \cdot x(t+\tau)]$

$= E[[A\cos\lambda t + B\sin\lambda t]$

$(A\cos(\lambda(t+\tau)) + B\sin(\lambda(t+\tau)))]$

$= E[A^2\cos\lambda t\ \cos\lambda(t+\tau) + B^2\sin\lambda t\ \sin\lambda(t+\tau)$

$+ AB\cos\lambda t\ \sin\lambda(t+\tau) + AB\sin\lambda t\ \cos\lambda(t+\tau)]$

$= \cos\lambda t\ \cos\lambda(t+\tau)\, E[A^2] +$

$\sin\lambda t.\ \sin\lambda(t+\tau)\, E[B^2] +$

$E[AB]\,[\cos\lambda t\ \sin\lambda(t+\tau) + \sin\lambda t\ \sin\lambda(t+\tau)]$

$= \cos\lambda t\ \cos\lambda(t+\tau)\,\varphi +$

$$= \varphi\left[\cos\lambda t \cos\lambda(t+\tau) + \sin\lambda t \sin\lambda(t+\tau)\right]$$

$$= \varphi\left[\cos(\lambda t - (\lambda(t+\tau))\right]$$

$$= \varphi\left[\cos(\lambda t - \lambda t - \lambda\tau\right]$$

$= \varphi \cos\lambda\tau$, free from t,

$X(t)$ is WSS.

5. Two random processes $X(t)$ and $Y(t)$ are given by

$$X(t) = A\cos\omega t + B\sin\omega t.$$

$$Y(t) = B\cos\omega t - A\sin\omega t.$$

Show that, $X(t)$ and $Y(t)$ are jointly WSS, If A and B are uncorrelated R.V with zero mean and the same variances with and ω is constant.

Solution:.

$$X(t) = A\cos\omega t + B\sin\omega t$$

$$Y(t) = B\cos\omega t - A\sin\omega t.$$

$$E[A] = E[B] = 0.$$

$$Var(A) = Var(B) = \sigma^2$$

$$\Rightarrow E[A^2] = E[B^2] = \sigma^2$$

A and B are uncorrelated $\Rightarrow E[AB] = 0.$

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(i) $X(t)$ is WSS

$E[X(t)] = E[A\cos\omega t + B\sin\omega t]$

$= E[A]\cos\omega t + E[B]\sin\omega t$

$= 0$, constant.

$R_{XX}(\tau) = E[X(t)\cdot X(t+\tau)]$

$= E\left[(A\cos\omega t + B\sin\omega t)(A\cos(\omega(t+\tau)) + B\sin(\omega(t+\tau)))\right]$

$= E\left[A^2\cos\omega t\cos(\omega(t+\tau)) + B^2\sin\omega t \sin(\omega(t+\tau)) + AB\left[\cos\omega t\sin(\omega(t+\tau)) + \sin\omega t\cos(\omega(t+\tau))\right]\right]$

$= E[A^2]\cos\omega t\cos(\omega(t+\tau)) + E[B^2]\sin\omega t\sin(\omega(t+\tau)) + E[AB]\left[\cos\omega t\sin(\omega(t+\tau)) + \sin\omega t\cos(\omega(t+\tau))\right]$

$= \sigma^2\left[\cos\omega t\cos(\omega(t+\tau)) + \sin\omega t\sin(\omega(t+\tau))\right] + 0$

$= \sigma^2(\cos(\omega t - \omega(t+\tau)))$

$= \sigma^2(\cos(\omega t - \omega t - \omega\tau))$

$= \sigma^2(\cos(-\omega\tau))$

$= \sigma^2\cos\omega\tau$, free from t,

$X(t)$ is WSS.

(ii) $Y(t)$ is WSS.

$E[Y(t)] = E[B \cos \omega t - A \sin \omega t]$

$= E[B] \cos \omega t - E[A] \sin \omega t.$

$= 0$, constant,

$R_{YY}(\tau) = E[Y(t) \cdot Y(t+\tau)]$

$= E[(B \cos \omega t - A \sin \omega t)(B \cos(\omega(t+\tau)) - A \sin(\omega(t+\tau))]$

$= E[B^2 \cos \omega t \cos(\omega(t+\tau)) + A^2 \sin \omega t \sin(\omega(t+\tau))$

$- AB \cos \omega t \sin(\omega(t+\tau)) - AB \sin \omega t \cos(\omega(t+\tau))]$

$= E[B^2] \cos \omega t \cos(\omega(t+\tau)) +$
$E[A^2] \sin \omega t \sin(\omega(t+\tau)) -$
$E[AB] [\cos \omega t \sin(\omega(t+\tau)) + \sin \omega t \cos(\omega(t+\tau))]$

$= \sigma^2 [\cos \omega t \cos(\omega(t+\tau)) + \sin \omega t \sin(\omega(t+\tau))]$

$= \sigma^2 (\cos(\omega t - \omega(t+\tau))]$

$= \sigma^2 (\cos(\cancel{\omega t} - \cancel{\omega t} - \omega\tau)]$

$= \sigma^2 [\cos(-\omega\tau)]$

$= \sigma^2 \cos \omega\tau$

$R_{YY}(t) = \sigma^2 \cos \omega\tau$, free from t.

$R_{XY}(\tau) = E[X(t) \cdot Y(t+\tau)]$

$= E[(A\cos\omega t + B\sin\omega t)(B\cos(\omega(t+\tau)) - A\sin(\omega(t+\tau)))]$

$= E[B^2 \sin\omega t \cdot \cos(\omega(t+\tau)) - A^2 \cos\omega t \cdot \sin(\omega(t+\tau))$

$+ AB \cos\omega t \cos(\omega(t+\tau)) - AB (\sin\omega t \sin(\omega(t+\tau))]$

$= \sigma_A^2 (\sin\omega t \cos(\omega(t+\tau)) - \cos\omega t \sin\omega(t+\tau))$

$+ 0$

$= \sigma^2 [\sin(\omega t - \omega t - \omega\tau)]$

$= \sigma^2 [\sin(-\omega\tau)]$

$= -\sigma^2 \sin\omega\tau$, free from t,

b. If $X(t) = Y\cos t + Z\sin t$ $\forall t$, where Y and Z are independent binary random variables each of which assumes the values $(-1, +2)$ with probabilities $2/3$ and $1/3$ respectively. Prove that $X(t)$ is WSS.

Solution:

Given

$X(t) = Y\cos t + Z\sin t$

Y	-1	2
$P(Y)$	$2/3$	$1/3$

Z	-1	2
$P(Z)$	$2/3$	$1/3$

$= E[Y^2] \cos t \cos(t+\tau) + E[Z^2] \sin t \sin(t+\tau)$

$+ E[YZ] [\cos t \sin(t+\tau) + \sin t \cos(t+\tau)]$

$= 2 [\cos t \cos(t+\tau) + \sin t \sin(t+\tau)] + 0$

$= 2 (\cos(t - t - \tau))$

$= 2 \cos(-\tau)$

$= 2 \cos\tau.$ free from t.

$X(t)$ is WSS.

Ergodicity:

A random process $X(t)$ is said to be ergodic, if its ensemble averages (statistical averages (ie) mean, autocorrelation) are equal to appropriate time averages.

If $X(t)$ is a random process, then $\frac{1}{2T}\int_{-T}^{T} X(t)\, dt$ is called time average of $X(t)$ over $(-T, T)$ and denoted by $\overline{X_T}$

$$\overline{X_T} = \frac{1}{2T}\int_{-T}^{T} X(t)\, dt.$$

If the random process $X(t)$ has a constant mean, as $T \to \infty$, then $x(t)$ is said to be mean ergodic.

Problem procedure:

Step 1: Find $\overline{x_T}$

Step 2: Find $E[\overline{x_T}]_T$

Step 3: $Var(\overline{x_T}) = \frac{1}{T}\int_{-T}^{T} C_{xx}(\tau) \left(1 - \frac{|\tau|}{T}\right) d\tau$

where

$$C_{xx}(\tau) = E[x(t)\ x(t+\tau)] - E[x(t)]\,E[x(t+\tau)]$$

Step 4: $\lim_{T \to \infty} Var(\overline{x_T}) = 0$

Correlation ergodic:

$x(t)$ is correlation ergodic,

if $\overline{Z}_T = \frac{1}{2T}\int_{-T}^{T} x(t+\tau)\ x(t)\ dt = R(\tau)$

as limit $T \to \infty$

7. If WSS process $x(t)$ is given by $x(t) = 10\cos(100t + \theta)$ where θ is uniformly distributed over $(-\pi, \pi)$. Prove that $x(t)$ is correlation ergodic.

Solution:

$$f(\theta) = \frac{1}{b-a} = \frac{1}{\pi+\pi} = \frac{1}{2\pi}$$

$$R_{xx}(\tau) = E[x(t) \cdot x(t+\tau)]$$

$$= E[10\cos(100t+\theta) \cdot 10\cos(100(t+\tau)+\theta)]$$

$$= \frac{100}{2} E\left[\cos(200t+100\tau+2\theta)+\cos(100\tau)\right]$$

$$= 50\, E\left[\cos(200t+100\tau+2\theta)\right]+E\left[\cos(100\tau)\right]$$

Consider,

$$E\left[\cos(200t+100\tau+2\theta)\right]$$

$$= \int_{-\pi}^{\pi} \cos(200t+100\tau+2\theta) \cdot \frac{1}{2\pi}\, d\theta$$

$$= \frac{1}{2\pi} \cdot 2\int_{0}^{\pi} \cos(200t+100\tau+2\theta) \cdot d\theta$$

$$= \frac{1}{\pi}\left[\frac{\sin(200t+100\tau+2\theta)\cdot 2}{2}\right]_{0}^{\pi}$$

$$= 0$$

Sub in (i)

$$R_{xx}(\tau) = 50\{0+\cos 100\tau\}$$

$$= 50 \cos 100\tau$$

$$\overline{Z}_T = \frac{1}{2T}\int_{-T}^{T} X(t+\tau)\; X(t)\,.\,dt = R(\tau).$$

$$= \frac{1}{2T}\int_{-T}^{T} 10\cos(100t+\theta)\; 10\cos(100(t+\tau)+\theta)\,dt$$

$$= \frac{100}{2T}\int_{-T}^{T} \cos(100t+\theta)\,.\,\cos(100t+100\tau+\theta)\,dt$$

$$= \frac{50}{T}\int^{T} \frac{1}{2}\left[\cos(200t+100\tau+2\theta)+\cos(-100\tau)\right]$$

$Var\, \bar{x}_T = 2/3$

$$\lim_{T \to \infty} Var\, \bar{x}_T = \frac{2}{3} \neq 0$$

$X(t)$ is not mean ergodic.

Consider 2 random Variable process,

$X(t) = 3\cos(\omega t + \theta)$ $\quad Y(t) = 2\cos(\omega t + \theta - \pi/2)$

where θ is a random Variable uniformly distributed in $(0, 2\pi)$. Prove that $\sqrt{R_{xx}(0) \cdot R_{yy}(0)} \geq |R_{xy}(\tau)|$

Solution,

Given.

$X(t) = 3\cos(\omega t + \theta)$

$Y(t) = 2\cos(\omega t + \theta - \pi/2)$

θ is uniformly distributed in $(0, 2\pi)$

$$f(\theta) = \frac{1}{b-a} = \frac{1}{2\pi - 0} = \frac{1}{2\pi}$$

$$R_{xx}(\tau) = E[x(t) \cdot x(t+\tau)]$$

$$= E[3\cos(\omega t + \theta) \cdot 3\cos(\omega(t+\tau) + \theta)]$$

$$= 9E[\cos(\omega t + \theta) \cdot \cos(\omega t + \omega\tau + \theta)]$$

$$= \frac{9}{2} E[\cos(2\omega t + \omega\tau + 2\theta) + \cos(-\omega\tau)]$$

$= 9 \; E[\cos(2\ldots$

$$= \frac{9}{2} E\left[\cos(\omega t+\omega\tau+2\theta)\right] + \frac{9}{2} E\left[\cos(\omega\tau)\right]$$

$$= \frac{9}{2}\int_0^{2\pi} \cos(2\omega t+\omega\tau+2\theta)\cdot\frac{1}{2\pi}\,d\theta + \frac{9}{2}\cos\omega\tau$$

$$= \frac{9}{4\pi}\underbrace{\left[\frac{\sin(2\omega t+\omega\tau+2\theta)}{2}\right]_0^{2\pi}}_{0} + \frac{9}{2}\cos\omega\tau$$

$$= \frac{9}{2}\cos\omega\tau$$

$$R_{XX}(0) = \frac{9}{2}\cos\omega(0)$$

$$= \frac{9}{2}$$

$$R_{YY}(\tau) = E\left[Y(t)\cdot Y(t+\tau)\right]$$

$$= E\left[2\cos(\omega t+\theta-\pi/2)\cdot 2\cos(\omega t+\omega\tau+\theta-\pi/2)\right]$$

$$= 4E\left[\cos(\omega t+\theta-\pi/2)\cdot\cos(\omega t+\omega\tau+\theta-\pi/2)\right]$$

$$= \frac{4}{2}E\left[\cos(2\omega t+\omega\tau+2\theta-\pi) + \cos(-\omega\tau)\right]$$

$$= 2E\left[\cos(2\omega t+\omega\tau+2\theta-\pi)\right] + 2E\left[\cos(\omega\tau)\right]$$

$$= 2\int_0^{2\pi}\cos(2\omega t+\omega\tau+2\theta+\pi)\frac{1}{2\pi}\,d\theta + 2\cos\omega\tau$$

$$= 2\left[\sin(2\omega t+\omega\tau+2\theta+\pi)\right]^{2\pi}$$

$$R_{YY}(0) = 2\cos w(0)$$

$$= 2$$

$$R_{XY}(\tau) = E[X(t) \cdot Y(t+\tau)]$$

$$= E[3\cos(wt+\theta) \cdot 2\cos(wt+w\tau+\theta-\pi/2)]$$

$$= 6E[\cos(wt+\theta) \cdot \cos(wt+w\tau+\theta-\pi/2)]$$

$$= 6E[\cos(2wt+w\tau+2\theta-\pi/2) + \cos(-w\tau+\pi/2)]$$

$$= 6E[\cos(2wt+w\tau+2\theta-\pi/2)] +$$

$$6E[\cos(\overset{\pi/2-w\tau}{\cancel{w\tau-\pi/2}})]$$

$$= 6\int_0^{2\pi} \cos(2wt+w\tau+2\theta-\pi/2) \cdot \frac{1}{2\pi}\, d\theta$$

$$+6\cos(\overset{\pi/2-w\tau}{\cancel{w\tau-\pi/2}})$$

$$= 6E[\cos(wt+\theta) \cdot \sin(wt+w\tau+\theta)]$$

$$= \frac{6}{2}E[\sin(2wt+w\tau+2\theta) + \sin(w\tau)]$$

$$= \frac{6}{2}E[\sin(2wt+w\tau+2\theta)] + \frac{6}{2}E[\sin w\tau]$$

$$= \frac{6}{2}\int^{2\pi} \sin(2wt+w\tau+2\theta) \cdot \frac{1}{2\pi}\, d\theta + \frac{6}{2}\sin w\tau$$

$$= \frac{6}{2}\left[\frac{\cos(2\omega t+\omega\tau+2\theta)}{2 \cdot 2\pi}\right]_0^{2\pi} + \frac{6}{2}\sin\omega\tau$$

$$= \frac{6}{2}\left[\frac{\cos(2\omega t+\omega\tau+2\pi)}{2 \quad 2\pi} - \cos(0)\right] + \frac{6}{2}\sin\omega\tau$$

$$= 3\sin\omega\tau$$

$$R_{XY}(\tau) = 3\sin\omega\tau$$

$$R_{XX}(0) \cdot R_{YY}(0) = \frac{9}{2} \cdot 2 = 9$$

$$\sqrt{R_{XX}(0) \cdot R_{YY}(0)} = \sqrt{9}$$

$$= 3$$

$$R_{XY}(\tau) = |3\sin\omega\tau| \leq 3$$

$$R_{XY}(\tau) \leq \sqrt{R_{XX}(0) \cdot R_{YY}(0)}.$$

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Markov process

Future depends only upon the Present but not on past.

If for all n, $P[X_n = a_n / X_{n-1} = a_{n-1}$

$$P[X_n = a_n / X_{n-1} = a_{n-1} \dots X_0 = a_0] = P[X_n = a_n / X_{n-1} = a_{n-1}]$$

$\{X_n\}$ then the process $\{X_n\}$, $n = 0, 1, 2 \dots$ is called Markov chain.

(i) $a_1, a_1, \dots a_n$ are called States.

(ii) $P[X_n = a_j / X_{n-1} = a_i]$ is called one step

(iii) $P[X_n = a_j / X_0 = a_i]$ is called 'n' step transistion probability from State a_i to a_j.

Note 1:

The tpm of a margkov Change is a Stochastic matrix since $P_{ij} \geq 0$ and $\sum P_{ij} = 1$ (ie) Sum of all the elements of row of the

Note 2:

A Stochastic matrix 'p' is said to be a regular matrix, if all the entries of P^m (Possible integer m) are Positive.

Note 3:

A homogeneous markov chain is said to be regular, if its tpm is regular

Note 4:

If $P_{ij}^{m} > 0$, for some 'n' and $\forall$ i and j, that every state can be reached from every other state. Here, markov chain is said to be irreducable.

Note 5:

The period di of a return state, i is defined as the greatest Common divisor of all m, such that $P_{ij}^{m} > 0$. State i is said to be periodic with period di, if $d_i > 1$, and a periodic if $d_i = 1$

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Note 6 :

A non-null persistent and Apeariodic state is called ergodic.

Note 7 :

If a markov chain is irreduceable, all its states are of the same time.

if a markov chain is finite irreduceable, all its states are non-null persistant.

Note 8 :

Steady state, probability distribution or Stationary state distribution of the markov chain is $\pi p = \pi$

Note 9 :

To find irreduceable nature ; $P^2, P^3, P^4 \ldots$ and note all $P_{ij} > 0$, at some "P^n".

To find period type ; called the Powers of P, where $P_{ii} > 0$, and find gcd of powers

To find steady state ; find $\pi p = \pi$

Find the nature of the states of the markov chain if the tpm

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}\end{array} \longrightarrow \text{future.}$$

$\downarrow$ Present

Solution,

Given.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^2 = P \cdot P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$P^3 = P \cdot P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^4 = P \cdot P^3 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

Markov chain is introducable and finite.

$\Rightarrow$ All states are non-null Persistant.

i) $P_{11}^{(2)} > 0$, $P_{11}^{(4)} > 0$

$\Rightarrow \gcd\{2, 4 \ldots\} = 2$

$\Rightarrow$ state (0 is Period 2)

$P_{22}^{(2)} > 0$, $P_{22}^{(4)} > 0$

$\gcd\{2, 4, \ldots\} = 2$

$\Rightarrow$ state 1 is Period 2

$P_{33}^{(2)} > 0$, $P_{33}^{(4)} > 0 \ldots$

$\gcd\{2, 4 \ldots\} = 2$

$\Rightarrow$ State 2 is period 2.

Here all states are periodic with period 2.

Here all states are non-null persistent and periodic

$\Rightarrow$ All States are ergodic

Three Boys A, B, C are throwing a ball to each other, A always throws the ball to B. and B always throws the ball to C. But C is just as likely to throw the Ball to B. as to A. Find ~~the~~ tpm and Classify the states.

Solution:

The tpm is

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

$$P^2 = P.P = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P^3 = P.P^2 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$P^4 = P^2.P^2 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$P^5 = P^2.P^3 = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$$

The markov chain is introducable & finite

All states, non-null persistant

$P_{11}^{(3)} > 0, P_{11}^{(5)} > 0$

$\gcd\{3, 5 \ldots\} = 1$

states 'A' is Period 1.

$P_{22}^{(2)} > 0, \quad P_{22}^{(3)} > 0, \quad P_{22}^{(4)} > 0 \ldots$

$\gcd\{2, 3, 4, \ldots\} = 1$

state 'B' is Period 1

$P_{33}^{(2)} > 0, \quad P_{33}^{(3)} > 0, \quad P_{33}^{(4)} > 0 \ldots$

$\gcd\{2, 3, 4 \ldots\} = 1$

states 'C' is period 1

All states are Aperiodic

All states are ergodic.

~~A man drives a car (or) cadges a train to go to office each day. He never goes two days in a row by train but if the drives one day then the next day he is just likely to drive again as he is travel by train. Now suppose that the first day of the weak. The the first day ... drove to~~

The markov chain is introducable
& finite

All states non-null persistant.

$P_{11}^{(3)} > 0$, $P_{11}^{(5)} > 0$

$gcd\{3, 5 \dots\} = 1$

states 'A' is Period 1.

$P_{22}^{(2)} > 0$, $P_{22}^{(3)} > 0$, $P_{22}^{(4)} > 0 \dots$

$gcd\{2, 3, 4, \dots\} = 1$

state 'B' is Period 1

$P_{33}^{(2)} > 0$, $P_{33}^{(3)} > 0$, $P_{33}^{(4)} > 0 \dots$

$gcd\{2, 3, 4 \dots\} = 1$

states 'C' is Period 1

All states are Aperiodic

All states are ergodic.

~~A man drives a car (or) cadges a train to go to office each day. He never goes two days in a row by train but if the drives one day then the next day he is just likely to drive again as he is travel by train. Now suppose that the first day of the weak. The ... drove to~~

The transition probability matrix of a Markov chain $\{X_n\} = 1,2,3$ having three states 1, 2 and 3 is P=

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$ and the initial

distribution is $p^{(0)} = [0.7, \ 0.2 \ \ 0.1]$

Find (i) $P[X_2 = 3]$

(ii) $P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$

Solution:

Given:

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$P^2 = P.P = \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

(i) $P[X_2 = 3] = \sum_{i=1}^{3} [P[X_2 = 3 / X_0 = i]. P[X_0 = i]$

$= P[X_2 = 3 / X_0 = 1] . P[X_0 = 1] +$

$P[X_2 = 3 / X_0 = 2] \ P[X_0 = 2] +$

$P[X_2 = 3 / X_0 = 3] \ P[X_0 = 3]$

$= P_{13}^{(2)} \ P[X_0 = 1] + P_{23}^{(2)} \ P[X_0 = 2] +$

$P_{33}^{(2)} \ P[X_0 = 3]$

$= (0.26 \times 0.7) + (0.34 \times 0.2) +$

$= 0.182 + 0.068 + 0.029$

$= 0.279.$

(ii) $P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2] =$

$P[X_3 = 2 / X_2 = 3, X_1 = 3, X_0 = 2] \; P[X_2 = 3, X_1 = 3, X_0 = 2]$

$= P[X_3 = 2 / X_2 = 3] \; P[X_2 = 3 / X_1 = 3, X_0 = 2] \; P[X_1 = 3, X_0 = 2]$

$= P[X_3 = 2 / X_2 = 3] \; P[X_2 = 3 / X_1 = 3] \; P[X_1 = 3 / X_0 = 2] \; P[X_0 = 2]$

$= P_{32}^{(1)} \cdot P_{33}^{(1)} \cdot P_{23}^{(1)} \cdot P[X_0 = 2]$

$= (0.4)(0.3)(0.2)(0.2)$

$= 0.0048.$

~~A State i is said to be recurrent, if the returned to State i, is certain, $F_{ii} = 1$, (or) Transient.~~

~~if the return to State i is uncertain $F_{ii} < 1$~~

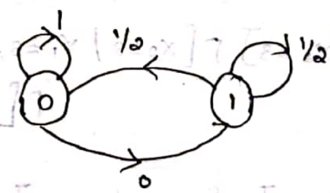
Consider a markov chain with a States $\{0,1\}$. and $P=\begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$

State (i) Draw the tpm diagram.

(ii) Is the State - 0 recurrent?

(iii) Is the State - 1 is transient?

Solution:

(i) 

(ii) State - 0 is recurrent

It returns to zero with probability 1,

(iii) State - 1 is Transient.

It returns to 1 with probability $1/2$.

Poisson process:

If $x(t)$ represents the number of occurence of a certain event in $(0,t)$, then discrete random process $x(t)$ is called the poisson process.

(i) $P(1 \text{ occurrence in } (t, t+\Delta t)] = \lambda\Delta t + o(\Delta t)$

(ii) $P[0 \text{ occurrence in } (t, t+\Delta t)] = (1-\lambda\Delta t) + o(\Delta t)$

(iii) $P[2 \text{ occurrence in } (t, t+\Delta t)] = o(\Delta t)$.

(iv) $x(t)$ is independent

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$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad n=0,1,2,\ldots$$

Second Order Probabity function of a homogeneous poisson process.

$$P[X(t_1)=n, X(t_2)=n_2] = P[X(t_1)=n_1]\, P[X(t_2)=n_2 / X(t_1)=n_1]$$

$$P[X(t_1)=n, X(t_2)=n_2] = \frac{e^{-\lambda t_1}(\lambda t_1)^n}{n_1!} \cdot \frac{e^{-\lambda(t_2-t_1)}\{\lambda(t_2-t_1)\}^{n_2-n_1}}{(n_2-n_1)!}$$

$n_2 \geq n_1$

Third Order probability function of a homogeneous poisson process:

$$P[X(t_1)=n, X(t_2)=n_2, X(t_3)=n_3] = \frac{e^{-\lambda t_3}\lambda^{n_3}.t_1^{n_1}(t_2-t_1)^{n_2-n_1}.(t_3-t_2)^{n_3-n_2}}{n!\,(n_2-n_1)!\,(n_3-n_2)!} \quad n_3 \geq n_2 \geq n_1$$

Mean of a poisson process:

$$\text{Mean} = E[X(t)] = \sum_{n=0}^{\infty} n\, P_n(t)$$

$$= \sum_{n=0}^{\infty} n.\, e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

$$= 0 + \sum_{n=1}^{\infty} e^{-\lambda t}\cdot\frac{(\lambda t)^n}{(n-1)!}$$

$$= e^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{(n-1)!}$$

$$= e^{-\lambda t}\Big[\lambda(t)\ (\lambda t)^2.\ (\lambda t)^3 \quad \Big]$$

Auto co-variance of the poisson process:

$$C_{xx}(t_1,t_2) = R_{xx}(t_1,t_2) - E[x(t_1)] \cdot E[x(t_2)]$$

$$= \lambda^2(t_1t_2) + \lambda t_1 - \lambda t_1 \cdot \lambda t_2$$

$$= \lambda^2(\cancel{t_1t_2}) + \lambda(t_1) - \lambda^2(\cancel{t_1t_2})$$

$$= \lambda(t_1)$$

$$\boxed{C_{xx}(t_1,t_2) = \lambda \min(t_1,t_2)}$$

Correlation Coefficient of the poisson process:.

$$\rho_{xx}(t_1,t_2) = \frac{C_{xx}(t_1,t_2)}{\sqrt{Var(x(t_1))} \cdot \sqrt{Var(x(t_2))}}$$

$$= \frac{\lambda(t_1)}{\sqrt{\lambda t_1} \cdot \sqrt{\lambda t_2}}$$

$$= \frac{\cancel{\lambda} t_1}{\cancel{\lambda}\sqrt{t_1t_2}}$$

$$= \sqrt{t_1/t_2} \quad t_1 \le t_2.$$

$$\boxed{\rho_{xx}(t_1,t_2) = \sqrt{\frac{t_1}{t_2}}, \quad t_1 \le t_2}$$

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Property 1:

Poisson process is a markov process:

Let us take the Conditional Probability distribution of $X(t_3)$ given the Past values of $X(t_2)$ and $X(t_1)$. Assume that $t_3 > t_2 > t_1$ and $n_3 > n_2 > n_1$

Consider $P[X(t_3) = n_3 / X(t_2) = n_2, X(t_1) = n_1]$

$$= \frac{P[X(t_3) = n_3, X(t_2) = n_2, X(t_1) = n_1]}{P[X(t_1) = n_1, X(t_2) = n_2]}$$

$$= \frac{e^{-\lambda t_3} \cdot \lambda^{n_3} \cdot t_1^{n_1} (t_2 - t_1)^{n_2 - n_1} (t_3 - t_2)^{n_3 - n_2}}{n_1! (n_2 - n_1)! (n_3 - n_2)!} \div \frac{e^{-\lambda t_2} \lambda^{n_2} t_1^{n_1} (t_2 - t_1)^{n_2 - n_1}}{n_1! (n_2 - n_1)!}$$

$$= \frac{e^{-\lambda t_3} \lambda^{n_3} \cdot \cancel{t_1^{n_1}} (t_2 - t_1)^{n_2 - n_1} (t_3 - t_2)^{n_3 - n_2}}{\cancel{n_1!} \cancel{(n_2 - n_1)!} (n_3 - n_2)!} \times \frac{\cancel{n_1!} \cancel{(n_2 - n_1)!}}{e^{-\lambda t_2} \lambda^{n_2} \cancel{t_1^{n_1}} \cancel{(t_2 - t_1)^{n_2 - n_1}}}$$

$$= \frac{e^{-\lambda t_3} \lambda^{n_3} (t_3 - t_2)^{n_3 - n_2}}{(n_3 - n_2)! \, e^{-\lambda t_2} \lambda^{n_2}}$$

$$= \frac{e^{-\lambda t_3} \cdot \lambda^{n_3} (t_3 - t_2)^{n_3 - n_2} \, e^{\lambda t_2} \lambda^{-n_2}}{(n_3 - n_2)!}$$

$$= \frac{e^{-\lambda(t_3-t_2)} \lambda^{(n_3-n_2)} (t_3-t_2)^{n_3-n_2}}{(n_3-n_2)!}$$

∴ Thus poisson process is a markov process.

Property 2:

Sum of two independent poisson Process is a poisson process.

Let $X(t) = X_1(t) + X_2(t)$.

$$P[X(t)=n] = \sum_{r=0}^{n} P[X_1(t)=r]\, P[X_2(t)=n-r]$$

$$= \sum_{r=0}^{n} e^{-\lambda_1 t} \cdot \frac{(\lambda_1 t)^r}{r!} \cdot e^{-\lambda_2 t} \frac{(\lambda_2 t)^{n-r}}{(n-r)!}$$

$$= e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \left[\sum_{r=0}^{n} \frac{(\lambda_1 t)^r}{r!} \cdot \frac{(\lambda_2 t)^{n-r}}{(n-r)!} \right]$$

$$= e^{-(\lambda_1+\lambda_2)t} \left[\sum_{r=0}^{n} \frac{\lambda_1^r t^r}{r!} \times \frac{\lambda_2^{n-r} \cdot t^{n-r}}{(n-r)!} \right]$$

$$= e^{-(\lambda_1+\lambda_2)t} \sum_{r=0}^{n} \frac{t^n \cdot n!}{n!\, r!\, (n-r)!} \cdot \lambda_1^r \lambda_2^{n-r}$$

$$= e^{-(\lambda+\lambda_2)t} \sum_{r=0}^{n} \frac{t^n}{n!} nC_r\, \lambda_1^r \lambda_2^{n-r}$$

$$= e^{-(\lambda_1+\lambda_2)t} \cdot t^n \sum \lambda_1^r \lambda_2^{n-r} \cdot nC_r$$

$$\left({}^nc_x \, p^x q^{n-x} \right) = (p+q)^n$$

$$\therefore P[X(t)=n] = e^{-(\lambda_1+\lambda_2)t} \frac{t^n}{n!} (\lambda_1+\lambda_2)^n$$

$$= e^{-(\lambda_1+\lambda_2)t} \cdot \frac{[(\lambda_1+\lambda_2)t]^n}{n!}$$

Thus $X_1(t) + X_2(t)$ is a poisson process.

Property 3:

Difference of two independant poisson process is not a poisson process.

Proof:

Let $X(t) = X_1(t) - X_2(t)$.

$$E[X(t)] = E[X_1(t) - X_2(t)]$$

$$= \lambda_1 t - \lambda_2 t$$

$$E[X(t)] = (\lambda_1 - \lambda_2) t$$

$$E[X^2(t)] = E\left[[X_1(t) - X_2(t)]^2\right]$$

$$= E[X_1^2(t) + X_2^2(t) - 2X_1(t) \cdot X_2(t)]$$

$$= E[X_1^2(t)] + E[X_2^2(t)] - 2E[X_1(t)] \cdot E[X_2(t)]$$

$$= (\lambda_1^2 t^2 + \lambda_1 t) + (\lambda_2^2 t^2 + \lambda_2 t) - 2\lambda_1 t \lambda_2 t$$

$= (\lambda_1+\lambda_2)t + (\lambda_1^2+\lambda_2^2 - 2\lambda_1\lambda_2 t)^2$

$= (\lambda_1+\lambda_2)t + (\lambda_1-\lambda_2)^2 t^2$

$[X_1(t) - X_2(t)]$ is not a poisson process.

Property 4:

The inter arrival time of a poisson process

The interval between two successive occurrences of a poisson process with parameter λ has an exponential distribution with mean $1/\lambda$.

Proof:

Let E_i and E_{i+1} be the two consecutive events

Let T be the interval b/w E_i & E_{i+1}

$P[T>t] = P$[no event occurs in the interval length 't']

$= P[X(t)=0]$

$P[X(t)=n] = \dfrac{e^{-\lambda t}(\lambda t)^n}{n!}$

$= \dfrac{e^{-\lambda t}(\lambda t)^0}{0!}$

$= e^{-\lambda t}$

$F(t) = P(T \leq t) = 1 - P(T>t)$

$$f(t) = F'(t) = -e^{-\lambda t}(-\lambda$$
$$= \lambda e^{-\lambda t}$$

$f(t) = \lambda e^{-\lambda t}$ is a pdf of an exponential distribution with mean $1/\lambda$

If the no. of occurrences of an event E in an interval of length t, is a Poisson process X(t) with parameter λ and if each occurrence of E, has a constant probability P. being recorded and the recordings are independent of each other, then the ~~no.~~ number N(t) of the recorded occurrences in 't' is also a poisson process with parameter λp

Solution:

P[n(t)]

$$P[N(t)=n] = \sum_{r=0}^{\infty} P[E \text{ occurs } (n+r) \text{ times in } t \text{ and } n \text{ of them are recorded}]$$

$$= \sum_{r=0}^{\infty} \left\{ \frac{e^{-\lambda t}(\lambda t)^{n+r}}{(n+r)!} \right\} \left\{ (n+r)C_n \, p^n q^{(n+r)-n} \right\}$$

$$= \sum_{r=0}^{\infty} \left\{ \frac{e^{-\lambda t} \lambda^n \lambda^r t^n t^r}{(n+r)!} \frac{(n+r)!}{(n+r-r)!\,r!} p^n q^{n+r-n} \right\}$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda t} \lambda^{r} \lambda^{n} t^{n} t^{r}}{\cancel{(n+r)!}} \cdot \frac{\cancel{(n+r)!}}{n!\,r!} p^{n} q^{r}$$

$$= \frac{e^{-\lambda t} (\lambda t p)^{n}}{n!} \sum_{r=0}^{\infty} \frac{(\lambda t q)^{r}}{r!}$$

$$= e^{-\lambda t} \frac{(\lambda t p)^{n}}{n!} \sum_{r=0}^{\infty} \frac{(\lambda t q)^{r}}{r!}$$

$$= e^{-\lambda t} \frac{(\lambda t p)^{n}}{n!} \cdot e^{\lambda t q}$$

$$= \frac{e^{-\lambda t(1-q)} \cdot (\lambda t p)^{n}}{n!}$$

$$= \frac{e^{-\lambda p t} (\lambda t p)^{n}}{n!}$$

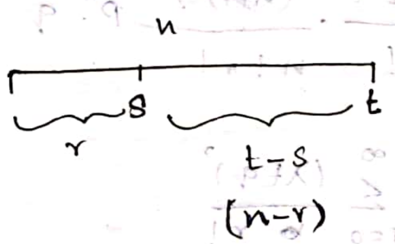
$N(t)$ follows poisson process with parameter λp.

(x.) If $X(t)$ is a poisson process, prove that,

$$P[X(s)=r / X(t)=n] = nC_r \left(\frac{s}{t}\right)^{r} \left(1-\frac{s}{t}\right)^{n-r}, \quad s<t$$

Proof:.

$$P[X(s)=r / X(t)=n] = \frac{P[X(s)=r \cap X(t)=n]}{P[X(t)=n]}$$



$$= \frac{P\left[X(s)=r \cap X(t-s)=n-r\right]}{P\left[X(t)=n\right]}$$

$$= \frac{\dfrac{e^{-\lambda s}(\lambda s)^{r}}{r!} \cdot \dfrac{e^{-\lambda(t-s)}\left(\lambda(t-s)\right)^{n-r}}{(n-r)!}}{\dfrac{e^{-\lambda t}(\lambda t)^{n}}{n!}}$$

$$= \frac{e^{-\lambda s}(\cancel{\lambda} s)^{r}}{r!} \cdot \frac{e^{-\lambda t}\, e^{\lambda s}\, \lambda^{n}\, \cancel{\lambda}\,(t-s)^{n-r}}{(n-r)!} \times \frac{n!}{e^{-\lambda t}\,\lambda^{n}\, t^{n}}$$

$$= \frac{nC_r\, s^{r}\, t^{n-r}\left(1-\frac{s}{t}\right)^{n-r}}{t^{n}}$$

$$= \frac{nC_r\, s^{r}\, \cancel{t^{n}}\, t^{-r}\left(1-\frac{s}{t}\right)^{n-r}}{\cancel{t^{n}}}$$

$$= nC_r \left(\frac{s}{t}\right)^{r}\left(1-\frac{s}{t}\right)^{n-r}$$

$$P\left[X(s)=r \,/\, X(t)=n\right] = nC_r\left(\frac{s}{t}\right)^{r}\left(1-\frac{s}{t}\right)^{n-r}$$

If customers arrive at a counter in accordance with the poisson process with a mean rate of 3 per minutes. Find the probability that the interval b/w two consecutive arrivals is

(a) More than 1 minute

(b) b/w 1 minute and 2 minute.

(c) 4 minute (or) less.

Solution:

$$P[T>1] = \int_{-\infty}^{\infty} \lambda e^{-\lambda t} dt$$

$$= \int_{1}^{\infty} 3 e^{-3t} dt$$

$$= 3\left[\frac{e^{-3t}}{-3}\right]_1^{\infty}$$

$$= -\left[e^{-\infty} - e^{-3}\right]$$

$$= -[0 - 0.4978]$$

$$= 0.4978.$$

$$P[1<T<2] = \int \lambda e^{-\lambda t} dt$$

$$= \int_1^2 3 e^{-3t} dt$$

$$= 3\left[\frac{e^{-3t}}{}\right]^2 dt$$

$$= -\left[e^{-6} - e^{-3}\right]$$

$$= -\left[0.002 - 0.0497\right]$$

$$= -\left[-0.0477\right]$$

$$= 0.0477$$

$$P[T \le 4] = \int_0^4 3e^{-3t}\,dt$$

$$= 3\left[\frac{e^{-3t}}{-3}\right]_0^4$$

$$= -\left[e^{-12} - e^{0}\right]$$

$$= -\left[0.00000614 - 1\right]$$

$$= -\left[-0.999\right]$$

$$= 0.999$$

Gaussian (Normal) process:

A real value random process $X(t)$ is said to be a Gaussian process (or) a normal process, if the random variables $X(t_1), X(t_2) \dots X(t_n)$ are jointly normal for a free $n = 1, 2 \dots$ and for any set of t_i's

Let $X(t)$ is a gaussian random process with $\mu[X(t)] = 10$ and $C_{XX}(t_1, t_2) = 16e^{-|t_1 - t_2|}$.

Find (i) $X(10) \leq 8$

(ii) $|X(10) - X(6)| \leq 4$

Solution:

$$C_{xx}(t_1, t_2) = R_{xx}(t_1, t_2) - E[X(t_1) \cdot X(t_2)]$$

$$C_{xx}(t_1, t_1) = R_{xx}(t_1, t_1) - E[X(t_1)]^2$$

$$= E[X^2(t_1)] - E[X(t_1)]^2$$

$$C_{xx}(t_1, t_1) = Var(X(t_1)) \quad \text{--- (1)}$$

Given:

$$C_{xx}(t_1, t_2) = 16e^{-|t_1 - t_2|}$$

Let $t = t_1 = t_2$.

~~$C_{xx}(t_1, t_1)$~~

$$C_{xx}(t, t) = 16e^{-|t-t|}$$

$$= 16e^{0}$$

$$C_{xx}(t, t) = 16 \quad \text{--- (2)}$$

Sub (2) in (1)

$$Var[X(t)] = C_{xx}(t, t) = 16$$

$\{X(t)\}$ is a random variable with Mean 10 and Variance 16.

(i) $P[X(10) \le 8]$

Let $Z = \frac{X-\mu}{\sigma}$

$$Z = \frac{X-10}{\sqrt{16}}$$

$$Z = \frac{X-10}{4}$$

$$P[X(10) \le 8) = P[Z \le \frac{8-10}{4})$$

$$= P[Z \le 0.5]$$

$$= 0.5 - P(0 < Z < 0.5)$$

$$= 0.5 - 0.1915$$

$$= 0.3085$$

(ii) $P[|X(10) - X(6)| \le 4]$

Let $U = X(10) - X(6)$.

$$E[U] = E[X(10)] - E[X(6)]$$

$E[X(t)] = \text{Mean} = 10$

$$= 10 - 10$$

$$E[U] = 0$$

$$E[U^2] = \cancel{E[X(10)] - E[X(6)]}$$

$$= E[[X(10) - X(6)]^2]$$

$$= E[X^2(10) + X^2(6) - 2X(10)X(6)]$$

$$= E[X^2(10)] + E[X^2(6)] - 2E[X(10)]E[X(6)]$$

$$= E[X^2(10)] + E[X^2(6) - \cancel{2E[X(10)]}$$

$$Var(U) = E[U^2] - [E[U]]^2$$
$$= E[x^2(10)] + E[x^2(6)] - 2\,cov\,(10,6).$$
$$= cov\,(10,10) + cov\,(6,6) - 2\,cov(10,6).$$
$$= 16+16-32\,e^{-4}$$
$$= \cancel{0}\; 32 - 32\,(0.0183)$$
$$= 32 - 0.5861$$
$$= 31.413$$
$$\sigma_U^2 = 31.413$$
$$\boxed{\sigma_U = 5.604}$$
$$P[|X(10) - X(6)| \leq 4] = P\{|U| \leq 4\}$$
$$= P[-4 \leq U \leq 4]$$
$$Z = \frac{U-\mu}{\sigma} = \frac{U-0}{5.604}$$
$U = -4$
$$Z = \frac{-4-0}{5.604} = -0.7136$$
$U = 4$
$$Z = \frac{4-0}{5.604} = 0.7136$$
$$P[-4 \leq U \leq 4] = P[-0.7136 < Z < 0.7136]$$
$$= P[\cancel{0.7136 < 0}] + $$

$= P[0 < z < 0.7136] + P[0 < z < 0.7136]$

$= 2P[0 < z < 0.7136]$

$= 2(0.611)$

$= 0.5222$

Suppose $X(t)$ is a normal process with mean $\mu(t) = 3$ and $C(t_1,t_2) = 4e^{-0.2|t_1-t_2|}$.

Find (i) $X(5) \leq 2$

(ii) $|X(8) - X(5)| \leq 1$

Solution:-

$C_{xx}(t_1, t_2) = R_{xx}(t_1, t_2) - E[X(t_1)\,X(t_2)]$

$C_{xx}(t_1, t_1) = R_{xx}(t_1, t_1) - E[X(t_1)]^2$

$= E[X^2(t_1)] - E[X(t_1)]^2$

$C_{xx}(t_1, t_1) = \text{var}(X(t_1))$ ——— (1)

Given

$C_{xx}(t_1, t_2) = 4e^{-0.2(t_1-t_2)}$

Put $t = t_1 = t_2$

$C_{xx}(t,t) = 4e^{-0.2(0)}$

$= 4$

$C_{xx}(t,t) = 4$ ——— (2)

substitute (2) in (1)

$\ldots] = C_{xx}(t,t) = 4$

mean 3 and variance 4.

(i) To find $P[x(5)] \leq 2$;

Let $z = \frac{x-\mu}{\sigma}$

$z = \frac{x-3}{\sqrt{4}}$

$z = \frac{x-3}{2}$

$P[x(5)] \leq 2 = P\left[z \leq \frac{2-3}{2}\right]$

$= P[z \leq -0.5]$

$= 0.5 - P[0 < z < 0.5]$

$= 0.5 - 0.1915$

$P[x(5)] \leq 2 = 0.3085$

(ii) $P[x(8) - x(5)] \leq 1$

Let $U = x(8) - x(5)$

$E[U] = E[x(8)] - E[x(5)]$

$= 3-3$

$= 0$

$E[U] = 0$.

$E[U^2] = E[[x(8) - x(5)]^2]$

$= E[x^2(8) + x^2(5) - 2x(8)x(5)]$

$= E[x^2(8)] + E[x^2(5)] - 2E[x(8)]E[x(5)]$

$= E[X^2(8)] + E[X^2(5)] - 2\,\mathrm{cov}(8,5)$

$= \mathrm{cov}(8,8) + \mathrm{cov}(5,5) - 2\,\mathrm{cov}(8,5)$

$= 4 + 4 - 8e^{-0.6}$

$= 8 - 8e^{-0.6}$

$= 8(1 - e^{-0.6})$

$\sigma_U^2 = 3.609$

$\sigma_U = 1.899$

$P[|X(8) - X(5)| \le 1] = P[|U| \le 1]$

$= P[-1 < U < 1]$

Let $z = \frac{U - \mu}{\sigma} = \frac{U - 0}{1.899}$

Put $U = -1 \Rightarrow z = \frac{-1}{1.899} = -0.5265$

Put $U = 1 \Rightarrow z = \frac{1}{1.899} = 0.5265$

$P[|X(8) - X(5)| \le 1] = P[-0.5265 < z < 0.5265]$

$= 2P[0 < z < 0.5625]$

$= 2(0.2019)$

$= 0.4038$

$P[|X(8) - X(5)| \le 1] = 0.4038$

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Random Telegram process:

Random Telegram process is a discrete random process $x(t)$, satisfies the following conditions

(1) $x(t)$ assumes only two values -1 and 1

(ii) $P[X(0)=1] = 1/2 = P[X(0)=-1]$

(iii) The no. of level transistions (or) flips ~~steps~~ $N(t)$ in the interval length t follows poisson process.

$$P[N(t)=r] = \frac{e^{-\lambda t}\lambda^{r}}{r!}, \quad r=1,2,3\ldots$$

Sine wave Process:

A sine wave random process is represented as $x(t) = A\sin(\omega t+\theta)$, where Amplitude A (or) Frequency ω (or) phase (or) any combination of these three may be removed.

For the sine wave process $X(t) = Y\cos\omega_0 t$, $-\infty < t < \infty$, ω_0 = Constant. The amplitude Y is a random variable with uniform distribution in the interval 0 to 1. Check whether the process is stationary or not.

Solution!

Given Y is uniformly distributed in the interval $(0,1)$

$$F(y) = \frac{1}{1-0} = 1$$

$$E[X(t)] = \int_{-\infty}^{\infty} x(t) \cdot F(y)\, dy$$

$$= \int_0^1 y \cos w_0 t \,(1)\, dy$$

$$= \cos w_0 t \int_0^1 y\, dy$$

$$= \cos w_0 t \left[\frac{y^2}{2}\right]_0^1$$

$$= \cos w_0 t \left(\tfrac{1}{2} - 0\right)$$

$$= \tfrac{1}{2} \cos w_0 t$$

Since the mean is time dependent

Thus the process is not stationary.

UNIT-IV

Correlation and Spectral densities:

Autocorrelation:

If the process $X(t)$ is either wide sense stationary (or) strict sense stationary then $E[X(t).\ X(t+\tau)]$ is a function of τ, denoded by $R(\tau)$ or $R_{XX}(\tau)$ or $R_X(\tau)$.

This fn/. $R_{XX}(\tau)$ is called the autocorrelation fn/. of the process $X(t)$.

(ie) $R_{XX}(\tau) = E[X(t).\ X(t+\tau)]$

Properties:

Property 1: The mean square value of the random process may be obtained from the auto correlation fn/. $R_{XX}(\tau)$, by putting $\tau = 0$.

Proof:

WKT

$$R_{XX}(\tau) = E[X(t).\ X(t+\tau)]$$

$$R_{XX}(0) = E[X(t).\ X(t)]$$

$$= E[X^2(t)]$$

(ie) $R_{XX}(0)$ is the mean square value. (ie) II moment of the random process.

Property 2:

$R_{xx}(\tau)$ is an even fn... of τ.

(ie) $R_{xx}(\tau) = R_{xx}(-\tau)$.

Proof:

WKT

$$R_{xx}(\tau) = E[x(t) \cdot x(t+\tau)]$$

$$R_{xx}(-\tau) = E[x(t) \cdot x(t-\tau)]$$

Put $(t-\tau) = p$

$$R_{xx}(-\tau) = E[x(p+\tau) \cdot x(p)]$$

$$= R_{xx}(\tau).$$

$$\therefore R_{xx}(-\tau) = R_{xx}(\tau).$$

Property 3:

The maximum value of $R_{xx}(\tau)$ is obtained at the point $\tau = 0$, (ie)

$$|R_{xx}(\tau)| \leq R_{xx}(0)$$

Proof:

Consider $E\{[x(t_1) \pm x(t_2)]^2\} \geq 0$

$$E[x^2(t_1) + x^2(t_2) \pm 2x(t_1) \, x(t_2)] \geq 0$$

$$E[x^2(t_1)] + E[x^2(t_2)] \pm 2E[x(t_1) \cdot x(t_2)] \geq 0$$

By property 1,

$$R_{xx}(0) + R_{xx}(0) \pm 2R_{xx}(t_1, t_2) \geq 0$$

$$2R_{xx}(0) \geq |2R_{xx}(\tau)|$$

Property 4:

If a random process $X(t)$ has no Periodic Components and its $X(t)$ is of non-zero mean, then $\lim_{|\tau|\to\infty} R_{xx}(\tau) = [E[X]]^2$

Property 5:

If $X(t)$ is periodic then its auto correlation fn/. is also periodic.

Property 6:

If the random process $Z(t) = X(t) + Y(t)$.

Where, $X(t)$ and $Y(t)$ are random process then $R_{ZZ}(\tau) = R_{XX}(\tau) + R_{YY}(\tau) + R_{XY}(\tau) + R_{YX}(\tau)$.

1. Given that the auto correlation fn/. for a Stationary ergodic process with no periodic components is $R_{xx}(\tau) = 25 + \frac{4}{1+6\tau^2}$. Find the mean and Variance of the process $X(t)$.

Solution:

Given.

$$R_{xx}(\tau) = 25 + \frac{4}{1+6\tau^2}$$

By using property 4:

$$(E[X])^2 = \bar{X}^2 = \lim_{\tau\to\infty} R_{xx}(\tau)$$

$$= \lim_{\tau\to\infty} 25 + \frac{4}{1+6\tau^2}$$

$$= 25 + \frac{4}{\infty}$$

$$= 25+0$$

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$\therefore E[X(t)] = 5$

By property 1;

$E[X^2(t)] = R_{XX}(0)$

$= 25 + \frac{4}{1+6(0)}$

$= 25 + 4$

$= 29$

$E[X^2(t)] = 29$

$Var[X(t)] = E[X^2(t)] - E([X(t)])^2$

$= 29 - 25$

$= 4$

$Var(X(t)) = 4$

2. Find the mean and Variance of a Stationary process whose autocorrelation function is $R_{XX}(\tau) = 18 + \frac{2}{6+\tau^2}$

3. Check whether the following fn's are Valid auto correlatio fnr.

$R_{XX}(\tau) = \frac{25\tau^2}{4+5\tau^2}$

$R_{XX}(\tau) = \tau^3 + \tau^2$

$R_{XX}(\tau) = \cos\tau + \frac{|\tau|}{T}$

2. Solution:

Given:

$$R_{xx}(\tau) = 18 + \frac{2}{6+\tau^2}$$

By using Property 4,

$$(E[X])^2 = \bar{X}^2 = \lim_{\tau\to\infty} R_{xx}(\tau)$$

$$= \lim_{\tau\to\infty} 18 + \frac{2}{6+\tau^2}$$

$$= 18 + \frac{2}{6+\infty}$$

$$= 18 + \frac{2}{\infty}$$

$$= 18 + 0$$

$$= 18$$

$$\bar{X}^2 = 18$$

$$\bar{X} = 4.24.$$

$$E[x(t)] = 4.24$$

By property 1:

$$E[x^2(t)] = R_{xx}(0)$$

$$= \cancel{25+\frac{4}{1+}} = 18 + \frac{2}{6+0}$$

$$= 18 + \frac{2}{6}$$

$$= 18.33$$

$$E[x^2(t)] = 18.33$$

$$Var[x(t)] = E[x^2(t)] - (E[x(t)])^2$$

$$= 18.33 - 18$$

$$= 0.33$$

$Var(x(t)) = 0.33$

3. Solution:

Given:

(i) $R_{xx}(\tau) = \dfrac{25\tau^2}{4+5\tau^2}$

$R_{xx}(-\tau) = \dfrac{25(-\tau)^2}{4+5(-\tau)^2}$

$= \dfrac{25\tau^2}{4+5\tau^2}$

$\therefore R_{xx}(\tau) = R_{xx}(-\tau)$

$\therefore R_{xx}(\tau)$ is a autocorrelation fn.

(ii) $R_{xx}(\tau) = \tau^3+\tau^2$

$R_{xx}(-\tau) = (-\tau)^3 + (-\tau)^2$

$= -\tau^3+\tau^2$ $\quad R_{xx}(\tau) \neq R_{xx}(-\tau)$.

$R_{xx}(\tau)$ is not a autocorrelation fn.

(iii) $R_{xx}(\tau) = \cos\tau + \dfrac{|\tau|}{T}$

$R_{xx}(-\tau) = \cos(-\tau) + \dfrac{|-\tau|}{T}$

$= \cos\tau + \dfrac{\tau}{T}$

$R_{xx}(\tau) = R_{xx}(-\tau)$.

$R_{xx}(\tau)$ is a autocorrelation fn.

4. Show that, a random process $X(t) = A \sin(\omega t+\varphi)$ where A and ω are constants, φ is a random variable uniformly distributed in $(0, 2\pi)$. Find the autocorrelation fn.. of the process.

Solution;

Given:

$$X(t) = A \sin(\omega t+\varphi).$$

φ is uniformly distributed in $(0, 2\pi)$

$$f(\varphi) = \frac{1}{b-a} = \frac{1}{2\pi - 0}$$

$$f(\varphi) = \frac{1}{2\pi}$$

$$R_{xx}(\tau) = E[x(t).\, x(t+\tau)]$$

$$= E[A \sin(\omega t+\varphi) .\, A \sin(\omega t+\omega\tau+\varphi)]$$

$$= A^2 E[\sin(\omega t+\varphi) .\, \sin(\omega t+\omega\tau+\varphi)]$$

$$= \frac{A^2}{2} E[\cos(-\omega\tau) - \cos(2\omega t+\omega\tau+2\varphi)]$$

$$= \frac{A^2}{2} E[\cos(\omega\tau) - \cos(2\omega t+\omega\tau+2\varphi)]$$

$$= \frac{A^2}{2} E[\cos(\omega\tau)] - \frac{A^2}{2} E[\cos(2\omega t+\omega\tau+2\varphi)]$$

$$= \frac{A^2}{2} \cos\omega\tau - \frac{A^2}{2} \int_0^{2\pi} \cos(2\omega t+\omega\tau+2\varphi) \underbrace{}_{f(\varphi)} d\varphi.$$

$$= \frac{A^2}{2} \cos\omega\tau - \frac{A^2}{2} \left[\frac{\sin(2\omega t+\omega\tau+2\varphi)}{2 . 2\pi}\right]_0^{2\pi} d\varphi.$$

$$= \frac{A^2}{2} \cos\omega\tau - \frac{A^2}{2} [0]$$

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Cross correlation:

Let $X(t)$ and $Y(t)$ be two random processes, then the cross correlation b/w them is defined as

$$R_{XY}(t,t+\tau) = E[X(t) \cdot Y(t+\tau)] = R_{XY}(\tau)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y;t,t+\tau)\, dx\, dy.$$

Property 1:

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

Proof:

$$R_{XY}(\tau) = E[X(t) \cdot Y(t+\tau)]$$

Consider,

$$R_{YX}(-\tau) = E[X(t-\tau)\; Y(t)]$$

Put $t-\tau = a \Rightarrow t = a+\tau$

$$R_{YX}(\tau) = E[X(a) \cdot Y(a+\tau)]$$

$$= R_{XY}(\tau).$$

Property 2:

If $X(t)$ and $Y(t)$ are two random process, $R_{XX}(\tau)$ and $R_{YY}(\tau)$ are their respective and correlation fn. then;

$$|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)\, R_{YY}(0)}$$

Consider,

$$E\left\{\left[\frac{X(t)}{\sqrt{R_{XX}(0)}} - \frac{Y(t)}{\sqrt{R_{YY}(0)}}\right]^2\right\} \geq 0$$

$$E\left[\frac{X^2(t)}{R_{XX}(0)} + \frac{Y^2(t)}{R_{YY}(0)} - 2\frac{X(t).Y(t)}{\sqrt{R_{XX}(0)\ R_{YY}(0)}}\right] \geq 0$$

$$E\left[\frac{X^2(t)}{R_{XX}(0)}\right] + E\left[\frac{Y^2(t)}{R_{YY}(0)}\right] - 2E\left[\frac{X(t).Y(t)}{\sqrt{R_{XX}(0).R_{YY}(0)}}\right] \geq 0.$$

$$\frac{1}{R_{XX}(0)} E\left[X^2(t)\right] + \frac{1}{R_{YY}(0)} E\left[Y^2(t)\right] - \frac{2}{\sqrt{R_{XX}(0).R_{YY}(0)}} E\left[X(t).Y(t)\right] \geq 0$$

$$\frac{R_{XX}(0)}{R_{XX}(0)} + \frac{R_{YY}(0)}{R_{YY}(0)} - \frac{2}{\sqrt{R_{XX}(0).R_{YY}(0)}} R_{XY}(\tau) \geq 0.$$

$$1+1 - \frac{2}{\sqrt{R_{XX}(0) R_{YY}(0)}} . R_{XY}(\tau) \geq 0$$

$$2 \geq 2 . \frac{R_{XY}(\tau)}{\sqrt{R_{XX}(0). R_{YY}0}}.$$

$$\sqrt{R_{XX}(0). R_{YY}(0)} \geq |R_{XY}(\tau)|$$

$$|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0). R_{YY}(0)}.$$

Hence proved.

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Property 3:

If $X(t)$ and $Y(t)$ are two random process then,

$$|R_{XY}(\tau)| \leq \frac{1}{2}\left[R_{XX}(0)+R_{YY}(0)\right]$$

Property 4:

If the random process $X(t)$ and $Y(t)$ are independent, then

$$R_{XY}(\tau) = E[X] \cdot E[Y].$$

$$R_{XY}(\tau) = E[X(t) \cdot Y(t+\tau)]$$

Property 5:

If the random process $X(t)$ and $Y(t)$ are of zero mean,

$$\lim_{\tau \to \infty} R_{XY}(\tau) = \lim_{\tau \to \infty} R_{YX}(\tau) = 0$$

Property 6:

The autocorrelation and cross correlation of two random processes $X(t)$ and $Y(t)$ can be expressed as a matrix called correlation matrix

$$R(\tau) = \begin{bmatrix} R_{XX}(\tau) & R_{XY}(\tau) \\ R_{YX}(\tau) & R_{YY}(\tau) \end{bmatrix}$$

then the random process $X(t)$ and $Y(t)$ are jointly WSS process.

Property 7:

Two random processes $X(t)$ and $Y(t)$ are said to be uncorrelated, if their cross correlation fn. is equal to the product of their means.

$$R_{XY}(\tau) = E[X(t)] \cdot E[Y(t+\tau)]$$

5. Two random processes $X(t)$ and $Y(t)$ are given by $X(t) = A\cos(\omega t+\theta)$, $Y(t) = A\sin(\omega t+\theta)$ where, A and ω are constants and θ is a uniform random variable over $(0, 2\pi)$. Find the cross correlation fn..

Solution:

Given:

$X(t) = A\cos(\omega t+\theta)$

$Y(t) = A\sin(\omega t+\theta)$

θ is uniform random variable,

$$f(\theta) = \frac{1}{b-a} = \frac{1}{2\pi - 0} = \frac{1}{2\pi}$$

$$f(\theta) = \frac{1}{2\pi}$$

$$R_{XY}(\tau) = E[X(t) \cdot Y(t+\tau)]$$

$$= E\left[A\cos(\omega t+\theta) \cdot A\sin\overset{(\omega t+\omega\tau+\theta)}{\cancel{(\omega t+\theta)}}\right]$$

$$= A^2 E\left[\cos(\omega t+\theta), \sin(\omega t+\omega\tau+\theta)\right]$$

$$= \frac{A^2}{\ } \left[\sin(2\omega t+\omega\tau+\theta) \cdot \cos(\omega t+\theta)\right]$$

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$$= \frac{A^2}{2} E\left[\sin(2\omega t+\omega\tau+2\theta) + \sin(-\omega\tau)\right]$$

$$= \frac{A^2}{2} E\left[\sin(2\omega t+\omega\tau+2\theta) + \sin\omega\tau\right]$$

$$= \frac{A^2}{2} E\left[\sin(2\omega t+\omega\tau+2\theta)\right] + E\left[\sin\omega\tau\right]$$

$$= \frac{A^2}{2}\sin\omega\tau + \frac{A^2}{2}\int_0^{2\pi} \sin(2\omega t+\omega\tau+2\theta)\cdot\frac{1}{2\pi}\,d\theta$$

$$= \frac{A^2}{2}\sin\omega\tau + \frac{A^2}{4\pi}\left[\frac{-\cos(2\omega t+\omega\tau+2\theta)}{2}\right]_0^{2\pi}$$

$\cos(4\pi+\theta) = \cos(2\pi+\theta) = \cos(\theta)$

$$= \frac{A^2}{2}\sin\omega\tau + \frac{A^2}{8\pi}\left[-\cos(2\omega t+\omega\tau+4\pi) + \cos(2\omega t+\omega\tau)\right]$$

$$= \frac{A^2}{2}\sin\omega\tau + \frac{A^2}{8\pi}\left[-\cancel{\cos(2\omega t+\omega\tau)} + \cancel{\cos(2\omega t+\omega\tau)}\right]$$

$$= \frac{A^2}{2}\sin\omega\tau$$

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Power Spectral Density

The power spectral density $S_{xx}(\omega)$ of a continuous time random process $X(t)$ is defined as the fourier transform of $R_{xx}(\tau)$, $S_{xx}(\omega) = -\infty$ to ∞

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) \cdot e^{-i\omega\tau} \, d\tau \quad \text{——} (1)$$

$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) \, e^{i\omega\tau} \, d\omega \quad \text{——} (2)$$

Eqn (1) and (2) are known as the Wiener-Khinchine relation.

Property:

(i) For a WSS, random process, prove spectral density at (zero) 0 frequency gives the area under the graph of autocorrelation.

(ii) The mean square value of a WSS process is equal to the total area under the graph of the spectral density.

(iii) The PSD of a real valued random process is an even fn/.. of frequency (or) The spectral density fn/.. of a real random process is an even fn/..

(iv) A WSS, random process has a non-

(v) The spectral density and the autocorrelation fn/.. of a real WSS process form a fourier cosine transform pair.

6. The PSD of a WSS process is given by $S(\omega) = \begin{cases} b/a\,(a-|\omega|) & |\omega| \le a \\ 0 & |\omega| \ge a \end{cases}$

Find the autocorrelation fn/.. of a process:

Solution:

Given:

$$S(\omega) = \begin{cases} \frac{b}{a}(a-|\omega|) & |\omega| \le a \\ 0 & |\omega| \ge a \end{cases}$$

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-a}^{a} \frac{b}{a}(a-|\omega|)\, e^{i\omega\tau}\, d\omega$$

$$= 2.\frac{1}{2\pi}\int_{0}^{a} \frac{b}{a}(a-\omega)\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{\pi}\,\frac{b}{a}\int_{0}^{a}(a-\omega)\, e^{i\omega\tau}\, d\omega.$$

$u = a-\omega$ $\qquad$ $v = e^{i\omega\tau}$

$u' = -\omega$ $\qquad$ $v_1 = \frac{e^{i\omega\tau}}{i\tau}$

$u'' = -1$ $\qquad$ $v_2 = \frac{e^{i\omega\tau}}{i^2\tau^2}$

$u''' = 0$

$$= \frac{b}{\pi a} \int_0^a (a-\omega)(\cos \omega \tau + i \sin \omega \tau)\, d\omega$$

$$= \frac{b}{\pi a} \int_0^a (a-\omega) \cos \omega \tau \, . d\omega$$

$u = a-\omega$ $\quad v = \cos \omega \tau$

$u' = -\omega$ $\quad v_1 = \frac{\sin \omega \tau}{\tau}$

$u'' = -1$

$v_2 = -\frac{\cos \omega \tau}{\tau^2}$

$u = a-\omega$ $\quad v = \cos \omega \tau$

$u' = -1$ $\quad v_1 = \frac{\sin \omega \tau}{\tau}$

$u'' = 0$

$v_2 = -\frac{\cos \omega \tau}{\tau^2}$

$$= \frac{b}{\pi a} \left[(a-\omega) \frac{\sin \omega \tau}{\tau} - \frac{\cos \omega \tau}{\tau^2} \right]_0^a$$

$$= \frac{b}{\pi a} \left[\left[0 - \frac{\cos a\tau}{\tau^2} \right] - \left[\frac{a \sin(0)}{\tau} - \frac{\cos(0)}{\tau^2} \right] \right]$$

$$= \frac{b}{\pi a} \left[-\frac{\cos a\tau}{\tau^2} + \frac{1}{\tau^2} \right]$$

$$= \frac{b}{a\tau^2 \pi} \left[1 - \cos a\tau \right] \qquad 1 - \cos \theta = 2 \sin^2 \theta/2$$

$$= \frac{b}{a\tau^2 \pi} \, 2 \sin^2 a\tau/2$$

7) The autocorrelation fn. of a WSS process is given by $R(\tau) = \alpha^2 e^{-2\lambda|\tau|}$. Determine the PSD of the process.

Solution:

Given, $R_{xx}(\tau) = \alpha^2 e^{-2\lambda|\tau|}$

Let $\alpha^2 = a$

$2\lambda = b$

$\therefore R_{xx}(\tau) = a e^{-b|\tau|}, \quad b>0$

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} a e^{-b|\tau|} \cdot e^{-i\omega\tau} \cdot d\tau$$

In $(-\infty, 0) \Rightarrow |\tau| = -\tau$

$(0, \infty) \Rightarrow |\tau| = \tau$

$$S_{xx}(\omega) = \int_{-\infty}^{0} a e^{-b(-\tau)} e^{-i\omega\tau} d\tau + \int_{0}^{\infty} a e^{-b\tau} e^{-i\omega\tau} d\tau$$

$$= a\int_{-\infty}^{0} e^{b\tau} \cdot e^{-i\omega\tau} d\tau + a\int_{0}^{\infty} e^{-b\tau} \cdot e^{-i\omega\tau} d\tau$$

$$= a\int_{-\infty}^{0} e^{(b-i\omega)\tau} d\tau + a\int e^{-(b+i\omega)\tau} \cdot d\tau$$

$$= a\left[\frac{e^{(b-i\omega)x}}{b-i\omega}\right]_{-\infty}^{0} + a\left[\frac{e^{-(b+i\omega)x}}{-(b+i\omega)}\right]_{0}^{\infty}$$

$$= \frac{a}{b-i\omega}\left[e^{0} - e^{-\infty}\right] - \frac{a}{b+i\omega}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{a}{b-i\omega}\left[1-0\right] - \frac{a}{b+i\omega}\left[0-1\right]$$

$$= \frac{a}{b-i\omega} + \frac{a}{b+i\omega}$$

$$= a\left[\frac{1}{b-i\omega} + \frac{1}{b+i\omega}\right]$$

$$= a\left[\frac{b+\cancel{i\omega} + b-\cancel{i\omega}}{b^2+\omega^2}\right]$$

$$= a\left[\frac{2b}{b^2+\omega^2}\right]$$

$$= \frac{2ab}{b^2+\omega^2}$$

$$S_{xx}(\omega) = \frac{2ab}{b^2+\omega^2}$$

$$S_{xx}(\omega) = \frac{2\alpha^2 2\lambda}{(2\lambda)^2+\omega^2}$$

$$\boxed{S_{xx}(\omega) = \frac{4\alpha^2\lambda}{4\lambda^2+\omega^2}}$$

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8. The PSD of a WSS process is given by $S(\omega) = \begin{cases} 1, & |\omega| \leq \omega_0 \\ 0 & \text{Otherwise.} \end{cases}$

Find the autocorrelation fn. of a process.

Solution:-

Given,

$$S(\omega) = \begin{cases} 1 & |\omega| \leq \omega_0 \\ 0 & \text{otherwise} \end{cases}$$

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} 1 \cdot e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\omega_0} (\cos\omega\tau + i\sin\omega\tau)\, d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\omega_0} \cos\omega\tau\, d\omega$$

$$= \frac{1}{\pi}\left[\frac{\sin\omega\tau}{\tau}\right]_{0}^{\omega_0}$$

$$= \frac{1}{\pi}\left[\frac{\sin\omega_0\tau}{\tau} - \frac{\sin(0)\tau}{\tau}\right]$$

$$= \frac{1}{\pi}\,\frac{\sin\omega_0\tau}{\tau}$$

9. Find PSD for the stationary process $X(t)$ with autocorrelation fn/. $R_{XX}(\tau) = a e^{-b|\tau|}$, $b>0$.

(or) $R_{XX}(\tau) = \sigma^2 e^{-\alpha|\tau|}$

10. The autocorrelation of the random binary transmission is given by $R_{XX}(\tau) = \begin{cases} 1-\frac{|\tau|}{T}, & |\tau| \le T \\ 0 & |\tau| \ge T \end{cases}$

Find PSD.

Solution:

Given:

$$R_{XX}(\tau) = \begin{cases} 1-\frac{|\tau|}{T} & |\tau| \le T \\ 0 & |\tau| \ge T \end{cases}$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-T}^{T} 1 - \frac{|\tau|}{T}\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-T}^{T} 1 - \frac{|\tau|}{T}\, (\cos\omega\tau - i\sin\omega\tau)\, d\tau$$

$$= \int_{-T}^{T} \left(1 - \frac{\tau}{T}\right) \cos\omega\tau\, d\tau$$

$$= \int_{-T}^{T} \left(\cos\omega\tau - \frac{\tau}{T}\cos\omega\tau\right) d\tau$$

$$= 2\int^{T} \cos\omega\tau\, d\tau - 2\int^{T} \tau$$

$$= 2\left[\frac{\sin \omega t}{\omega}\right]_0^T - \frac{2}{T}\int_0^T t \cos \omega t \, dt$$

$u = t \qquad v = \cos \omega t$

$u' = 1 \qquad v_1 = \frac{\sin \omega t}{\omega}$

$u'' = 0 \qquad v_2 = \frac{-\cos \omega t}{\omega^2}$

$$= 2\left[\frac{\sin \omega T}{\omega} - \frac{\sin(0)}{\omega}\right] - \frac{2}{T}\left[t\frac{\sin \omega t}{\omega} + \frac{\cos \omega t}{\omega^2}\right]_0^T$$

$$= \frac{2\sin \omega T}{\omega} - \frac{2}{T}\left[T\frac{\sin \omega T}{\omega} + \frac{\cos \omega T}{\omega^2} - \frac{\cos 0}{\omega^2}\right]$$

$$= \cancel{\frac{2\sin \omega T}{\omega}} - \cancel{\frac{2\sin \omega T}{\omega}} - \frac{2\cos \omega T}{T\omega^2} + \frac{2}{T\omega^2}$$

$$= \frac{2}{T\omega^2}\left[1 - \cos \omega T\right]$$

CROSS SPECTRAL DENSITY.

Let $R_{XY}(\tau)$ and $R_{YX}(\tau)$ be their Cross correlation fn.. then Cross Spectral densities are

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{-i\omega\tau}\, d\tau.$$

Property 1:

$$S_{YX}(\omega) = S_{XY}(-\omega)$$

Property 2:

Real part of $S_{XY}(\omega)$ is an even fn.. of ω

Property 3:

Imag part of $X_{XY}(\omega)$ is an odd fn.. of ω

Property 4:

$S_{XY}(\omega) = 0$, if $X(t)$ ~~and~~ are orthogonal

Property 5:

If $X(t)$ and $Y(t)$ are un-correlated,

$$S_{XY}(\omega) = E[X].\,E[Y].\,\delta(\omega).$$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$X(t)$ and $Y(t)$ are said to be uncorrelated.

11. The cross power spectrum of real random processes $X(t)$ and $Y(t)$ is given by,

$$S_{XY}(\omega) = \begin{cases} a + \frac{ib\omega}{\alpha} & , -\alpha < \omega < \alpha, \alpha > 0 \\ 0 & , \text{Otherwise} \end{cases}$$

Find the cross correlation fn..

Solution:

Given

$$S_{XY}(\omega) = \begin{cases} a + \frac{ib\omega}{\alpha} & , -\alpha < \omega < \alpha, \alpha > 0 \\ 0 & , \text{Otherwise} \end{cases}$$

Cross correlation fn.. is

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int \left(a + \frac{ib\omega}{\alpha}\right) e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\alpha}^{\alpha} a\, e^{i\omega\tau}\, d\omega + \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \frac{ib\omega}{\alpha}\, e^{i\omega\tau}\, d\omega$$

$$= \frac{a}{2\pi}\int_{-\alpha}^{\alpha} e^{i\omega\tau}\, d\omega + \frac{ib}{2\pi\alpha}\int_{-\alpha}^{\alpha} \omega\, e^{i\omega\tau}\, d\omega$$

$$= \frac{2a}{2\pi}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_0^{\alpha} + \frac{2ib}{2\pi\alpha}\int_0^{\alpha} \omega\, e^{i\omega\tau}\, d\omega$$

$$= \frac{a}{2\pi}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-\alpha}^{\alpha} + \frac{ib}{2\pi\alpha}\int_{-\alpha}^{\alpha} \omega e^{i\omega\tau}\, d\omega$$

$$= \frac{a}{i2\pi i}\left[e^{i\alpha\tau} - e^{-i\alpha\tau}\right] + \frac{ib}{2\pi\alpha}\left[\omega\cdot\frac{e^{i\omega\tau}}{i\tau} + \frac{e^{i\omega\tau}}{\tau^2}\right]_{-\alpha}^{\alpha}$$

$$= \frac{a}{2\pi i\tau}\left[e^{i\alpha\tau} - e^{-i\alpha\tau}\right] + \frac{ib}{2\pi\alpha}\left[\left[\alpha\frac{e^{i\alpha\tau}}{i\tau} + \frac{e^{i\alpha\tau}}{\tau^2}\right] - \left[-\alpha\frac{e^{-i\alpha\tau}}{i\tau} + \frac{e^{-i\alpha\tau}}{\tau^2}\right]\right]$$

$$= \frac{a}{2\pi i\tau}\left[2i\sin\alpha\tau\right] + \frac{ib}{2\pi\alpha}\left[\frac{\alpha}{i\tau}\left[e^{i\alpha\tau} + e^{-i\alpha\tau}\right] + \frac{1}{\tau^2}\left[e^{i\alpha\tau} - e^{-i\alpha\tau}\right]\right]$$

$$= \frac{a}{\pi\tau}\sin\alpha\tau + \frac{ib}{2\pi\alpha}\left[\frac{\alpha}{i\tau}2\cos\alpha\tau + \frac{1}{\tau^2}2i\sin\alpha\tau\right]$$

$$= \frac{a}{\pi\tau}\sin\alpha\tau + \frac{ib2\alpha}{2\pi\alpha i\tau}\cos\alpha\tau + \frac{ib2i}{2\pi\alpha\tau^2}\sin\alpha\tau$$

$$R_{xx}(\tau) = \frac{a}{\pi\tau}\sin\alpha\tau + \frac{b}{\dots}\cos\alpha\tau - \frac{b}{\dots}\sin\dots$$

4/4/18

12 If the cross correlation of two processes $X(t)$ and $Y(t)$ is $R_{XY}(t, t+\tau) =$

$\frac{AB}{2}\left[\sin w_0\tau + \cos w_0(2t+\tau)\right]$ where A and B, w_0 are constants. Find the cross power spectrum and time average

Solution:

The time average is given by

$$P_{XX} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{XY}(t, t+\tau)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left(\frac{AB}{2}\sin w_0\tau + \cos w_0(2t+\tau)\right) dt$$

$$\cancel{= \lim_{T\to\infty}\frac{AB}{2}\left[\lim_{T\to\infty}\int \sin w_0\tau\, dt\right.}$$

$$= \frac{AB}{2}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sin w_0\tau\, dt +$$

$$\frac{AB}{2}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos w_0(2t+\tau)\, dt$$

$$= \frac{AB}{2}\sin w_0\tau \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dt +$$

$$\frac{AB}{2}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos w_0(2t+\tau)\, dt$$

$$= \frac{AB}{2}\sin w_0\tau \lim_{T\to\infty}\frac{1}{2T}\left[T\right]_{-T}^{T} +$$

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$= \frac{AB}{2} \sin w_0 \tau \lim_{T \to \infty} \frac{1}{2T} [T+T] + 0$

$= \frac{AB}{2} \sin w_0 \tau \lim_{T \to \infty} \frac{1}{2T} \cdot 2T$

$= \frac{AB}{2} \sin w_0 \tau .$

Cross Power Spectrum is

$S_{XY}(w) = FT \text{ of } \frac{AB}{2} \sin w_0 \tau$

$$= \int_{-\infty}^{\infty} \frac{AB}{2} \sin w_0 \tau \, e^{-iw\tau} \, d\tau$$

$$= \frac{AB}{2} \int_{-\infty}^{\infty} \sin w_0 \tau \left(\cos w\tau - i \sin w\tau\right) d\tau$$

$$= \frac{AB}{2} \int_{-\infty}^{\infty} \sin w_0 \tau . \cos w_0 \tau \, d\tau \; -$$

$$i \frac{AB}{2} \int_{-\infty}^{\infty} \sin w_0 \tau . \sin w\tau \, d\tau .$$

$$= \frac{AB}{2} \int_{-\infty}^{\infty} \frac{1}{2} \sin(w_0 + w)\tau + \sin(w_0 + w)\tau \; -$$

$$i \frac{AB}{2} \int_{-\infty}^{\infty} \frac{1}{2} \cos(w_0 - w)\tau - \cos(w_0 + w)\tau$$

$\frac{AB}{}$ $\int^{\infty}$

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$$= \frac{AB}{4}\left[\int_{-\infty}^{\infty}\left[i\left(\cos(\omega+\omega_0)\tau - i\sin(\omega+\omega_0)\tau\right)\right.\right.$$

$$\left.\left. - i\left(\cos(\omega-\omega_0)\tau - i\sin(\omega-\omega_0)\tau\right)\right] d\tau\right.$$

$\sin(-\theta) = \sin\theta$

$$= \frac{ABi}{4}\int_{-\infty}^{\infty}\left(e^{-i(\omega+\omega_0)\tau} - e^{-i(\omega-\omega_0)\tau}\right) d\tau$$

$$= \frac{-iAB}{4}\int_{-\infty}^{\infty}\left(e^{-i(\omega-\omega_0)\tau} - e^{-i(\omega+\omega_0)\tau}\right) d\tau$$

$$= -\frac{iAB}{4}\int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)\tau} d\tau + \frac{iAB}{4}\int_{-\infty}^{\infty} e^{-i(\omega+\omega_0)\tau} d\tau.$$

$$= -\frac{iAB}{4}\left[2\pi\,\delta(\omega-\omega_0) - 2\pi\,\delta(\omega+\omega_0)\right]$$

$\delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau} d\tau$ is the

dirac - delta fn/.. Such that

$$\int_{-\infty}^{\infty}\delta(\omega)\,d\omega = 1$$

$$\therefore\ S_{xy}(\omega) = -\frac{i\pi AB}{2}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

13 If $X(t)$ and $Y(t)$ are uncorrelated random process, then find the power spectral density of Z, if $Z(t) = X(t)+Y(t)$. Also find the cross spectral density $S_{XZ}(\omega)$ and $S_{YZ}(\omega)$

Solution :

If $X(t)$ and $Y(t)$ are uncorrelated random process, their cross covariance $C_{XY}(t, t+\tau) = 0$

$$R_{XY}(t, t+\tau) - E[X(t)]\, E[Y(t)] = 0$$

$$R_{XY}[t, t+\tau] = E[X(t)] \cdot E[Y(t)]$$

$$R_{XY}(\tau) = E[X(t)] \cdot E[Y(t)]$$

Illy

$$R_{YX}(\tau) = E[Y(t)] \cdot E[X(t)]$$

$$= R_{XY}(\tau)$$

$$Z(t) = X(t) + Y(t).$$

$$R_{ZZ}(\tau) = E[Z(t) \cdot Z(t+\tau)]$$

$$= E\big[[X(t)+Y(t)][X(t+\tau)+Y(t+\tau)]\big]$$

$$= E[X(t) \cdot X(t+\tau) + X(t) \cdot Y(t+\tau) + Y(t) \cdot X(t+\tau) + Y(t) \cdot Y(t+\tau)]$$

$$= E[X(t) \cdot X(t+\tau)] + E[X(t) \cdot Y(t+\tau)] + E[Y(t) \cdot X(t+\tau)] + E[Y(t) \cdot Y(t+\tau)]$$

$$S_{ZZ}(\omega) = \int_{-\infty}^{\infty} R_{ZZ}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} \left(R_{XX}(\tau) + 2R_{XY}(\tau) + R_{YY}(\tau)\right) e^{-i\omega\tau} \cdot d\tau$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau + \int_{-\infty}^{\infty} 2R_{XY}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$+ \int_{-\infty}^{\infty} R_{YY}(\tau)\, e^{-i\omega\tau}\, d\tau.$$

$$S_{ZZ}(\omega) = S_{XX}(\omega) + 2S_{XY}(\omega) + S_{YY}(\omega).$$

Cross correlation function:

$$R_{XZ}(\tau) = E[X(t) \cdot Z(t+\tau)]$$

$$= E[X(t) \cdot X(t+\tau) + Y(t+\tau)]$$

$$= E[X(t) \cdot X(t+\tau) + X(t) \cdot Y(t+\tau)]$$

$$= E[X(t) \cdot X(t+\tau)] + E[X(t) \cdot Y(t+\tau)]$$

$$R_{XZ}(\tau) = R_{XX}(\tau) + R_{XY}(\tau).$$

$$S_{XZ}(\omega) = \int_{-\infty}^{\infty} R_{XZ}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} \left(R_{XX}(\tau) + R_{XY}(\tau)\right) e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} \cancel{R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau}$$

$$= S_{xx}(\omega) + S_{xy}(\omega)$$

$$S_{XZ}(\omega) = S_{xx}(\omega) + S_{xy}(\omega)$$

$$\lim_{T\to\infty} \frac{1}{2T} \cancel{\int E\left[|X_T(\omega)|^2\right]} = \cancel{S_{xx}(\omega)}.$$

||| ly $S_{YZ}(\omega) = S_{YY}(\omega) + S_{YX}(\omega)$

Find the mean square value of the process. PSD is b/w $S_{xx}(\omega) = \frac{\omega^2+2}{\omega^4+13\omega^2+36}$

Solution:

Given

$$S_{xx}(\omega) = \frac{\omega^2+2}{\omega^4+13\omega^2+36}$$

~~a=1~~ ~~b=13~~ ~~c=36~~

$$= \frac{\omega^2+2}{(\omega^2)^2+13\omega^2+36}$$

$$= \frac{\omega^2+2}{(\omega^2+9)(\omega^2+4)}$$

By partial fraction method.

$$\frac{\omega^2+2}{(\omega^2+9)(\omega^2+4)} = \frac{A}{\omega^2+9} + \frac{B}{\omega^2+4}$$

$$\frac{\omega^2+2}{(\omega^2+9)\cancel{(\omega^2+4)}} = \frac{A(\omega^2+4)+B(\omega^2+9)}{(\omega^2+9)\cancel{(\omega^2+4)}}$$

$$\omega^2+2 = A(\omega^2+4) + B(\omega^2+9)$$

$-2 = 5B$

$\boxed{B = -2/5}$

Put $\omega^2 = -9$

$-9+2 = A(-9+4) + B(-9+9)$

$-7 = A(-5)$

$7 = 5A$

$\boxed{A = 7/5}$

$$\therefore \frac{\omega^2+2}{(\omega^2+9)(\omega^2+4)} = \frac{7/5}{\omega^2+9} - \frac{2/5}{\omega^2+4}$$

$$S_{xx}(\omega) = \frac{7/5}{\omega^2+9} - \frac{2/5}{\omega^2+4}$$

$$R_{xx}(\tau) = F^{-1}[S_{xx}(\omega)]$$

$$= F^{-1}\left[\frac{7}{5}\cdot\frac{1}{\omega^2+9} - \frac{2}{5}\cdot\frac{1}{\omega^2+4}\right]$$

$$= \frac{7}{5}F^{-1}\left[\frac{1}{\omega^2+9}\right] - \frac{2}{5}F^{-1}\left[\frac{1}{\omega^2+4}\right]$$

$$\left[\because F^{-1}\left[\frac{2\alpha}{\omega^2+\alpha^2}\right] = e^{-\alpha|\tau|}\right]$$

$$= \frac{7}{5}F^{-1}\left[\frac{1}{6}\cdot\frac{2.3}{\omega^2+3^2}\right] - \frac{2}{5}F^{-1}\left[\frac{1}{4}\cdot\frac{2.2}{\omega^2+4}\right]$$

$$= \frac{7}{30}F^{-1}\left[\frac{6}{\omega^2+3^2}\right] - \frac{2}{20}F^{-1}\left[\frac{4}{\omega^2+2^2}\right]$$

$$R_{xx}(\tau) = \frac{7}{?}e^{-3|\tau|} - \frac{1}{?}e^{-2|\tau|}$$

$E[x^2(t)] = R_{xx}(0)$

$R_{xx}(0) = \frac{7}{30} e^{-3(0)} - \frac{1}{10} e^{-2(0)}$

$= \frac{7}{30} - \frac{1}{10}$

$= \frac{7-3}{30}$

$= \frac{4}{30}$

$= 2/15$

$E[x^2(t)] = 2/15$

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Unit - 5

Linear Systems with Random inputs

Linear System:

A System with functional relationship $f\{x(t)\}$ is linear, if, for any two inputs $x_1(t)$ and $x_2(t)$, the output of the System can be defined as $f\{(a_1 x_1(t) + a_2 x_2(t)\} = a_1 f\{x_1(t)\} + a_2 f\{x_2(t)\}$ where a_1 and a_2 are constants

Time invariance:

Time invariance is defined as a property of linear Systems that if the input is time Shifted by an amount τ, the corresponding Output will also be time shifted by the same amount.

(i.e) if $f\{x(t)\} = y(t)$ then

$$f\{x(t-\tau)\} = y(t-\tau), \quad -\infty < \tau < \infty$$

A system that does not meet the condition is called time Varying System.

Linear time invariant system:

Property : 1

Show that $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$ where $S_{XX}(\omega)$ and $S_{YY}(\omega)$ are the power spectral density functions of the input $X(t)$ and the output $Y(t)$ and $H(\omega)$ is the System transfer function.

Proof :

1. $Y(t) = \int^{\infty} h(u) X(t-u)\, du$

Property : 2

If the input $x(t)$ and its output $y(t)$ are related by $y(t) = \int_{-\infty}^{\infty} h(u) \cdot x(t-u)\, du$, then the system is linear time invariant system

Proof

First, we prove the linearity,

Consider, $x(t) = a_1 x_1(t) + a_2 x_2(t)$ —— (1)

then $y(t) = \int_{-\infty}^{\infty} h(u) \cdot x(t-u)\, du$.

$$= \int_{-\infty}^{\infty} h(u) [a_1 x_1(t-u) + a_2 x_2(t-u)] du$$

$$= a_1 \int_{-\infty}^{\infty} h(u) x_1(t-u)\, du + a_2 \int_{-\infty}^{\infty} h(u) x_2(t-u) du$$

$$= a_1 y_1(t) + a_2 y_2(t)$$

∴ The System is linear.

Now, we prove that the System is a time invariant System

Replace t by $t+k$

$$y(t) = \int_{-\infty}^{\infty} h(u) \; x[(t+k)-u]\, du$$

$$= y(t+k)$$

The System is time invariant.

Hence the System is linear time invariant System.

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Property : 3

If $\{x(t)\}$ is a WSS process and if $y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$, then $R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$

Proof:

Given $y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$ ——— (1)

WKT

$$R_{xy}(\tau) = E[\, x(t)\, y(t+\tau)\,]$$

$$= E\left[x(t) \int_{-\infty}^{\infty} h(u)\, x(t+\tau-u)\, du \right]$$

[by (1)]

$$= \int_{-\infty}^{\infty} E[\, x(t)\, x(t+\tau-u)\,]\, h(u)\, du$$

$t - t - \tau + u$
$- (\tau - u)$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau-u)\, h(u)\, du$$

[$\because$ $x(t)$ is WSS]

$$R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$$ [by convolution]

Property : 4

If $\{x(t)\}$ is a WSS process and if $y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$, then $R_{yy}(\tau) = R_{xy}(\tau) * h(-\tau)$ where $*$ denotes the convolution.

Proof :

Given $y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$ ——— (1)

$$R_{yy}(\tau) = E[\, y(t)\, y(t+\tau)\,]$$

$$= E\left[\int_{-\infty}^{\infty} h(u)\, x(t-u)\, y(t+\tau)\, du \right]$$

$$= \int_{-\infty}^{\infty} E[\, x(t-u)\, y(t+\tau)\, h(u)\, du\,]$$

$t-u-t-\tau$
$-(\tau+u)$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau+u)\, h(u)\, du.$$

Put $u = -\alpha$

$du = -d\alpha$

$$= \int_{\infty}^{-\infty} R_{xy}(\tau-\alpha)\, h(-\alpha)\, (-d\alpha)$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau-\alpha)\, h(-\alpha)\, d\alpha$$

$$= R_{xy}(\tau) * h(-\tau)$$

<u>Property 5</u>

If $\{x(t)\}$ is a WSS process and if $y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$ then $S_{xy}(\omega) = S_{xx}(\omega) * H(\omega)$

<u>Proof:</u>

$$y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$$

$$R_{xy}(\tau) = E[x(t)\, y(t+\tau)]$$

$$= E\left[x(t) \int_{-\infty}^{\infty} h(u)\, x(t+\tau-u)\, du\right]$$

$$= \int_{-\infty}^{\infty} E[x(t)\, x(t+\tau-u)]\, h(u)\, du$$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau-u)\, h(u)\, du$$

$$R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$$

Taking Fourier Transform

$= F[R_{xx}(\tau)] \, F[h(\tau)]$

$S_{xy}(\omega) = S_{xx}(\omega) \, H(\omega)$ [by defn of Spectrum]

---

1. Show that $\{x(t)\}$ is a WSS process then the output $\{y(t)\}$ is a WSS process.

Soln:

If the input to a time invariant, stable linear System is a WSS process, then the output will also be a WSS process

(i.e) To show that if $\{x(t)\}$ is a WSS process then the output $\{y(t)\}$ is a WSS process.

WKT the input and output are related by $y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$ —— (1)

$E[y(t)] = \int_{-\infty}^{\infty} h(u)\, E[x(t-u)]\, du$

$\because \{x(t)\}$ is a WSS process, Mean is Constant

(i.e) $E[x(t-u)]$ = constant

Hence $E[y(t)] = E[x'(t-u)] \int_{-\infty}^{\infty} h(u)\, du$

$= \overline{X_u} \int_{-\infty}^{\infty} h(u)\, du$

= a finite constant, independent of t

[$\because$ System is Stable]

$E[y(t)]$ = contant

Next to ST $R_{yy}(t, t+\tau)$ depends only on τ.

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)\, X(t-u_1)\, h(u_2)\, X(t+\tau-u_2)\, du_1\, du_2\right] \quad (2)$$

[by (1)]

$\because$ $\{X(t)\}$ is a WSS process.

$E[X(t-u_1)\, X(t+\tau-u_2)]$ is a function of τ,

Say $g(\tau)$.

$$(2) \Rightarrow R_{YY}(t, t+\tau) = g(\tau)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)\, h(u_2)\, du_1\, du_2$$

$= a$ function of τ

$\therefore$ The output $\{Y(t)\}$ is also WSS process.

---

(2) Find the Mean Square Value of the processes whose power Spectral density is as given below. $\dfrac{1}{\omega^4+10\omega^2+9}$. To find the Mean Square Value of the process, we can find its auto Correlation function and Substitute $\tau = 0$

Soln

$$S_X(\omega) = \frac{1}{\omega^4+10\omega^2+9}$$

$$= \frac{1}{(\omega^2+9)(\omega^2+1)} = \frac{A}{\omega^2+9} + \frac{B}{\omega^2+1}$$

$$\frac{A}{\omega^2+9} + \frac{B}{\omega^2+1} = \frac{A(\omega^2+1) + B(\omega^2+9)}{(\omega^2+9)(\omega^2+1)}$$

$$\frac{1}{(\omega^2+9)\cancel{(\omega^2+1)}} = \frac{A(\omega^2+1) + B(\omega^2+9)}{(\omega^2+9)\cancel{(\omega^2+1)}}$$

$$1 = A(\omega^2+1) + B(\omega^2+9)$$

Put $\omega^2 = -9$ | Put $\omega^2 = -1$

$1 = 0 + B(8)$

$$\frac{1}{(\omega^2+9)(\omega^2+1)} = \frac{-1/8}{\omega^2+9} + \frac{1/8}{\omega^2+1}$$

$$= \frac{1}{8}\left[\frac{1}{\omega^2+1} - \frac{1}{\omega^2+9}\right]$$

$R_{xx}(\tau)$ is fourier inverse transform of

$$\frac{1}{8}\left[\frac{1}{\omega^2+1} - \frac{1}{\omega^2+9}\right] \qquad \left[\because F^{-1}\left[\frac{2\alpha}{\alpha^2+\omega^2}\right] = e^{-\alpha|\tau|}\right]$$

$$R_{xx}(\tau) = F^{-1}\left\{\frac{1}{8}\left[\frac{1}{\omega^2+1} - \frac{1}{\omega^2+9}\right]\right\}$$

$$= \frac{1}{8}F^{-1}\left[\frac{1}{\omega^2+1}\right] - \frac{1}{8}F^{-1}\left[\frac{1}{\omega^2+3^2}\right]$$

$$= \frac{1}{8}\cdot\frac{1}{2}e^{-|\tau|} - \frac{1}{8}\cdot\frac{1}{6}e^{-3|\tau|}$$

$$= \frac{1}{16}e^{-|\tau|} - \frac{1}{48}e^{-3|\tau|}$$

The Mean square value is $R_{xx}(\tau)$ at $\tau = 0$.

$$R_{xx}(0) = \frac{1}{16} - \frac{1}{48} = \frac{2}{48} = \frac{1}{24}$$

---

(3) A system has an impulse response $h(t) = e^{-\beta t}U(t)$, find the power spectral density of the output $Y(t)$ corresponding to the input $X(t)$.

soln

Gn $X(t) \longrightarrow$ input process

$Y(t) \longrightarrow$ output process

WKT $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$ ——— (1)

.... the Fourier transform of the

The unit step function $U(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$

$$h(t) = \begin{cases} 0, & t < 0 \\ e^{-\beta t}, & t \geq 0 \end{cases}$$

$$\therefore H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt$$

$$= \int_{0}^{\infty} e^{-\beta t} e^{-i\omega t} dt$$

$$= \int_{0}^{\infty} e^{-(\beta+i\omega)t} dt$$

$$= \left[\frac{e^{-(\beta+i\omega)t}}{-(\beta+i\omega)} \right]_0^{\infty}$$

$$= -\frac{1}{\beta+i\omega} \left[e^{-(\beta+i\omega)t} \right]_0^{\infty}$$

$$= -\frac{1}{\beta+i\omega} [0-1]$$

$$= \frac{1}{\beta+i\omega}$$

$$|H(\omega)| = \frac{1}{|\beta+i\omega|} = \frac{1}{\sqrt{\beta^2+\omega^2}}$$

$$|H(\omega)|^2 = \frac{1}{\beta^2+\omega^2}$$

(1) $\Rightarrow S_{yy}(\omega) = \frac{1}{\beta^2+\omega^2} S_{xx}(\omega).$

Auto - correlation function of response :

$$R_{yy}(\tau) = R_{xx}(\tau) * h(-\tau) * h(\tau)$$

(4) **Cross correlation functions of input and output**

(i) $R_{xy}(\tau) = h(\tau) * R_{xx}(\tau)$

Proof:

The cross correlation fun: of $x(t)$ and $y(t)$ is

$R_{xy}(t, t+\tau) = E[x(t)\, y(t+\tau)]$ ——— ①

Now, $y(t+\tau) = h(t) * x(t+\tau)$

WKT
$$= \int_{-\infty}^{\infty} h(\varepsilon)\, x(t+\tau-\varepsilon)\, d\varepsilon \quad \text{— ②}$$

Sub ② in ①

$$R_{xy}(t, t+\tau) = E\left[x(t) \int_{-\infty}^{\infty} h(\varepsilon)\, x(t+\tau-\varepsilon)\, d\varepsilon\right]$$
$$= \int_{-\infty}^{\infty} E\{x(t)\, x(t+\tau-\varepsilon)\}\, h(\varepsilon)\, d\varepsilon \quad \text{— ③}$$

If $x(t)$ is WSS, ③ becomes.

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} R_{xx}(\tau-\varepsilon)\, h(\varepsilon)\, d\varepsilon$$

$$R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$$

III^{ly} $R_{yx}(\tau) = R_{xx}(\tau) * h(-\tau)$.

⑤ Consider a System with transfer function $\frac{1}{1+i\omega}$ An input Signal with auto correlation function $m\,\delta(\tau) + m^2$ is fed as input to the System. find the Mean and Mean square value of the input.

Soln

Gn $H(\omega) = \frac{1}{1+i\omega}$

& $R_{xx}(\tau) = m\,\delta(\tau) + m^2$

$S_x(\omega) = m + 2\pi m^2 \delta(\omega)$

WKT $S_y(\omega) = |H(\omega)|^2 S_x(\omega)$

$$= \left|\frac{1}{(1+i\omega)}\right|^2 [m + 2\pi m^2 \delta(\omega)]$$

WKT $S_{yy}(\omega) = S_{xx}(\omega) |H(\omega)|^2$ —— ①

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2|\tau|}\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{0} e^{2\tau} e^{-i\omega\tau}\, d\tau + \int_{0}^{\infty} e^{-2\tau} e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{0} e^{(2-i\omega)\tau}\, d\tau + \int_{0}^{\infty} e^{-(2+i\omega)\tau}\, d\tau$$

$$= \left[\frac{e^{(2-i\omega)\tau}}{(2-i\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-(2+i\omega)}}{-(2+i\omega)}\right]_{0}^{\infty}$$

$$= \left[\left(\frac{1}{2-i\omega}\right) - 0\right] + \left[0 - \frac{1}{-(2+i\omega)}\right]$$

$$= \frac{1}{2-i\omega} + \frac{1}{2+i\omega}$$

$$= \frac{2+i\omega+2-i\omega}{4+\omega^2}$$

$$S_{xx}(\omega) = \frac{4}{2^2+\omega^2}$$

$$H(\omega) = \frac{1}{\omega+2i} = \frac{1}{\omega+2i} \times \frac{\omega-2i}{\omega-2i} = \frac{\omega-2i}{\omega^2+2^2}$$

$$|H(\omega)| = \sqrt{\left(\frac{\omega}{\omega^2+4}\right)^2 + \left(\frac{2}{\omega^2+4}\right)^2} = \sqrt{\frac{\omega^2+4}{(\omega^2+4)^2}} = \frac{1}{\sqrt{\omega^2+4}}$$

$$① \Rightarrow S_{yy}(\omega) = \left(\frac{4}{4+\omega^2}\right)\left(\frac{1}{\omega^2+4}\right) = \frac{4}{(\omega^2+4)^2}$$

White Noise :

The term noise is used to designate unwanted signals that tend to disturb the [illegible] and processing of signals [illegible] we have

Shot Noise :

The discrete nature of electrons causes a signal disturbance called Shot Noise

Thermal Noise :

This noise is due to the random motion of free electrons in a conducting Medium such as a resistor.

White Noise (or) Gaussian Noise

The Noise analysis of communication systems is based on an idealized form of noise called white Noise

Power Spectral density of Thermal Noise :

The power spectral density of the noise current due to the free electrons is given by

$$S_i(\omega) = \left[\frac{2kTG\alpha^2}{\alpha^2+\omega^2}\right] = \frac{2kTG}{1+\left(\frac{\omega}{\alpha}\right)^2}$$

where k is the Boltzmann's constant
α is the average number of collision / second
T is the ambient temperature in degrees kelvin
G is the conductance of the conducting medium

Band limited White Noise :

Noise having a non-zero and constant spectral density over a finite frequency band and zero elsewhere is called band limited white Noise (i.e) if $\{N(t)\}$ is a band limited white noise then

$$S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & |\omega| \le \omega_B \\ 0, & \text{elsewhere} \end{cases}$$

(7) Consider a white Gaussian noise of zero mean and power spectral density $N_0/2$ applied ... RC filter whose transfer function

is $H(f) = \dfrac{1}{1+i2\pi fRC}$. find the autocorrelation function of the output random process.

Soln

Gn $H(f) = \dfrac{1}{1+i2\pi fRC}$

$|H(f)| = \dfrac{1}{|1+i2\pi fRC|} = \dfrac{1}{\sqrt{1+4\pi^2 f^2 R^2 C^2}}$

$|H(f)|^2 = \dfrac{1}{1+4\pi^2 f^2 R^2 C^2}$ ——— ①

Gn $S_{XX}(f) = \dfrac{N_0}{2}$ ——— ② [∵ the input is a white noise]

The power spectral densities of the input $\{X(t)\}$ and the output $\{Y(t)\}$ of a linear system are connected by

$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$ ——— ③

In the given problem the transfer function is expressed interms of the frequency f.

$S_{YY}(f) = |H(f)|^2 S_{XX}(f)$

$= \dfrac{1}{1+4\pi^2 f^2 R^2 C^2} \dfrac{N_0}{2}$ [by ① & ②]

$R_{YY}(\tau) = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} S_{YY}(f)\, e^{i\omega\tau}\, d\omega$

$= \dfrac{1}{2\pi} \int_{-\infty}^{\infty} \dfrac{1}{1+4\pi^2 f^2 R^2 C^2} \cdot \dfrac{N_0}{2}\, e^{i2\pi f\tau}\, d(2\pi f)$

[∵ $\omega = 2\pi f$]

$= \dfrac{1}{2\pi} \int_{-\infty}^{\infty} \dfrac{1}{1+4\pi^2 f^2 R^2 C^2}\, e^{i2\pi\tau f}\, 2\pi\, df$

$= \dfrac{N_0}{2} \int_{-\infty}^{\infty} \dfrac{1}{1+4\pi^2 f^2 R^2 C^2}\, e^{i2\pi\tau f}\, df$

$$R_{YY}(\tau) = \frac{N_0}{8\pi^2R^2C^2}\int_{-\infty}^{\infty}\frac{e^{i(2\pi\tau)f}}{\left(\frac{1}{2\pi RC}\right)^2+f^2}\,df \quad \text{--- (4)}$$

$$= \frac{N_0}{8\pi^2R^2C^2}\ \frac{\pi}{\left(\frac{1}{2\pi RC}\right)}\ e^{-|2\pi\tau|\left(\frac{1}{2\pi RC}\right)} \qquad \left[\because \int_{-\infty}^{\infty}\frac{e^{imx}}{a^2+x^2}dx = \frac{\pi}{a}e^{-|m|a}\right]$$

$$= \frac{N_0}{8\pi^2R^2C^2}\ 2\pi^2RC\ e^{-|2\pi\tau|\left(\frac{1}{2\pi RC}\right)}$$

$$R_{YY}(\tau) = \frac{N_0}{4RC}\ e^{-|\tau|/RC}$$

The Mean Square Value of $\{Y(t)\}$ is given by

$$E[Y^2(t)] = R_{YY}(0)$$

$$= \frac{N_0}{4RC}e^{\frac{-0}{RC}} = \frac{N_0}{4RC}e^{-0} = \frac{N_0}{4RC}(1)$$

$$= \frac{N_0}{4RC}$$

---

(8) If $\{X(t)\}$ is a band limited process such that $S_{XX}(\omega) = 0$, when $|\omega| > \sigma$, prove that $2[R_{XX}(0) - R_{XX}(\tau)] \le \sigma^2\tau^2 R_{XX}(0)$.

Soln:

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{i\tau\omega}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,(\cos\omega\tau + i\sin\omega\tau)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\cos\omega\tau\, d\omega$$

$$R_{XX}(0) - R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)(1-\cos\omega\tau)\,d\omega \qquad [\because X(t) \text{ is band limited}]$$

$$= \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)\, 2\sin^2\left(\frac{\omega\tau}{2}\right)d\omega \quad \text{--- (1)}$$

WKT $|\sin\theta| \le \theta$

$$2\sin^2\left(\frac{\omega\tau}{2}\right) \le \frac{\tau^2\omega^2}{2} \quad \text{——} \quad (2)$$

$$R_{XX}(0) - R_{XX}(\tau) \le \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)\,\frac{\tau^2\omega^2}{2}\,d\omega$$

[by (1)]

$$\le \frac{\sigma^2\tau^2}{4\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)\,d\omega$$

$$\le \frac{\sigma^2\tau^2}{4\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega$$

$$\le \frac{\sigma^2\tau^2}{\cancel{4}\pi}\cdot \cancel{2}\pi R_{XX}(0)$$

$$\le \frac{\sigma^2\tau^2}{2} R_{XX}(0).$$

---

(9) If $Y(t) = A\cos(\omega_0 t + \theta) + N(t)$, where A is a constant, θ is a random variable with a uniform distribution in $(-\pi, \pi)$ and $\{N(t)\}$ is a band limited Gaussian white noise with a power spectral density.

$$S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & \text{for } |\omega - \omega_0| < \omega_B \\ 0, & \text{elsewhere} \end{cases}$$

Find the power spectral density of $\{Y(t)\}$. Assume that $N(t)$ and θ are independent.

Soln

Gn $Y(t) = A\cos(\omega_0 t + \theta) + N(t)$

$$Y(t+\tau) = A\cos(\omega_0(t+\tau) + \theta) + N(t+\tau)$$

$$= A\cos[\omega_0 t + \omega_0\tau + \theta] + N(t+\tau)$$

$$Y(t)\,Y(t+\tau) = [A\cos(\omega_0 t + \theta) + N(t)][A\cos(\omega_0 t + \omega_0\tau + \theta) + N(t+\tau)]$$

$$= A^2\cos(\omega_0 t + \theta)\cos(\omega_0 t + \omega_0\tau + \theta)$$

$$\ldots)N(t+\tau) + A\cos(\omega_0 t + \omega_0\tau + \theta)N(t)$$

$$R_{YY}(t, t+\tau) = E[Y(t)\,Y(t+\tau)]$$

$$= E[A^2 \cos(\omega_0 t+\theta)\cos(\omega_0 t+\omega_0\tau+\theta) + A\cos(\omega_0 t+\theta)\,N(t+\tau) + A\cos(\omega_0 t+\omega_0\tau+\theta)N(t) + N(t)\,N(t+\tau)]$$

$$= A^2 E[\cos(\omega_0 t+\theta)\cos(\omega_0 t+\omega_0\tau+\theta)] + A E[\cos(\omega_0 t+\theta)\,N(t+\tau)] + A E[\cos(\omega_0 t+\omega_0\tau+\theta)\,N(t)] + E[N(t)\,N(t+\tau)]$$

$$= \frac{A^2}{2} E[2\cos(\omega_0 t+\omega_0\tau+\theta)\cos(\omega_0 t+\theta)] + A E[\cos(\omega_0 t+\theta)\,N(t+\tau)] + A E[\cos(\omega_0 t+\omega_0\tau+\theta)\,N(t)] + E[N(t)\,N(t+\tau)]$$

$$= \frac{A^2}{2} E[\cos(\omega_0 t+\omega_0\tau+\theta+\omega_0 t+\theta) + \cos(\omega_0 t+\omega_0\tau+\theta-\omega_0 t-\theta)] + A E[\cos(\omega_0 t+\theta)\,N(t+\tau)] + A E[\cos(\omega_0 t+\omega_0\tau+\theta)\,N(t)] + E[N(t)\,N(t+\tau)]$$

$$= \frac{A^2}{2} E[\cos(2\omega_0 t+2\theta+\omega_0\tau) + \cos\omega_0\tau] + A E[\cos(\omega_0 t+\theta)]\,E(N(t+\tau)) + A E[\cos(\omega_0 t+\omega_0\tau+\theta)]\,E[N(t)] + R_{NN}(\tau)$$

[$\because$ $N(t)$ is Stationary] —— (1)

An θ is uniformly distributed in $(-\pi, \pi)$

$$\therefore f(\theta) = \frac{1}{2\pi}, \quad -\pi < \theta < \pi$$

$$E[\cos(\omega_0 t+\theta)] = \int_{-\infty}^{\infty} \cos(\omega_0 t+\theta)\, f(\theta)\, d\theta$$

$$= \int_{-\pi}^{\pi} \cos(\omega_0 t+\theta)\,\frac{1}{2\pi}\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} [\cos\omega_0 t\cos\theta - \sin\omega_0 t\sin\theta]\, d\theta.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\omega_0 t \cos\theta \, d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\omega_0 t \sin\theta \, d\theta$$

$$= \frac{1}{2\pi}\cos\omega_0 t \int_{-\pi}^{\pi}\cos\theta \, d\theta - \frac{1}{2\pi}\sin\omega_0 t \int_{-\pi}^{\pi}\sin\theta \, d\theta$$

$$= \frac{1}{2\pi}\cos\omega_0 t \; 2\int_{0}^{\pi}\cos\theta \, d\theta - \frac{1}{2\pi}\sin\omega_0 t \int_{-\pi}^{\pi}\sin\theta \, d\theta$$

$$= \frac{1}{2\pi}\cos\omega_0 t \; (2)\int_{0}^{\pi}\cos\theta \, d\theta - \frac{1}{2\pi}\sin\omega_0 t \; (0)$$

$$= \frac{1}{\pi}\cos\omega_0 t \left[\sin\theta\right]_0^{\pi} - 0$$

$$= \frac{1}{\pi}\cos\omega_0 t \left[0-0\right]$$

$$= \frac{1}{\pi}$$

$$= 0 \quad \text{—— } ②$$

$$E[\cos(\omega_0 t + \omega_0 \tau + \theta)] = 0 \quad \text{—— } ③$$

$$E[\cos(2\omega_0 t + 2\theta + 2\omega_0 \tau)] = \int_{-\pi}^{\pi}\cos(2\omega_0 t + 2\theta + \omega_0\tau)\frac{1}{2\pi}d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(2\omega_0 t + 2\theta + \omega_0\tau)\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(2\omega_0 t + \omega_0\tau)\cos 2\theta - \sin(2\omega_0 t + \omega_0\tau)\sin 2\theta \, d\theta$$

$$= \frac{1}{2\pi}\cos(2\omega_0 t + \omega_0\tau)\int_{-\pi}^{\pi}\cos 2\theta \, d\theta - \frac{1}{2\pi}\sin(2\omega_0 t + \omega_0\tau)\int_{-\pi}^{\pi}\sin 2\theta \, d\theta$$

$$= \frac{1}{2\pi}\cos(2\omega_0 t + \omega_0\tau)\; 2\int_{0}^{\pi}\cos 2\theta \, d\theta - \frac{1}{2\pi}\sin(2\omega_0 t + \omega_0\tau)(0)$$

$$= \frac{1}{2\pi}\cos(2\omega_0 t + \omega_0\tau)\left[\frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\cos(2\omega_0 t + \omega_0\tau)\left[\sin 2\theta\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\cos(2\omega_0 t + \omega_0\tau)\left[0-0\right]$$

$$= \frac{1}{2\pi}\cos(2\omega_0 t + \omega_0\tau)$$

$$= 0 \quad \text{—— } ④$$

$$① \Rightarrow R_{YY}(t, t+\tau) = \frac{A^2}{2}\cos(\omega_0\tau) + R_{NN}(\tau)$$

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} \left[\frac{A^2}{2} \cos \omega_0 \tau + R_{NN}(\tau) \right] e^{-i\omega\tau} \, d\tau$$

$$= \frac{A^2}{2} \int_{-\infty}^{\infty} \cos \omega_0 \tau \, e^{-i\omega\tau} \, d\tau + \int_{-\infty}^{\infty} R_{NN}(\tau) \, e^{-i\omega\tau} \, d\tau$$

$$= \pi \frac{A^2}{2} \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right] + S_{NN}(\omega)$$

$$= \frac{\pi A^2}{2} \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right] + \frac{N_0}{2}$$

$\left[\because \text{Gn } S_{NN}(\omega) = \frac{N_0}{2} \right]$